A CHARACTERIZATION OF $C(X)$

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It is a classical fact that there exist harmonic functions u in the unit disk with conjugate harmonic function v such that u has continuous boundary values on the unit circumference, while v does not. Let us restate this fact as follows:

Denote by A_0 the algebra of functions analytic in $|z| < 1$ with continuous boundary values on $|z| = 1$ and write $\boldsymbol{Re}\, A_0$ for the space of all real parts of functions in A_0. Then we may say: there exists a harmonic function u in $|z| < 1$ with continuous boundary values such that u does not lie in $\boldsymbol{Re}\, A_0$. On the other hand, u is certainly a uniform limit of functions in $\boldsymbol{Re}\, A_0$ on $|z| = 1$, for all finite real trigonometric polynomials on $|z| = 1$ are in $\boldsymbol{Re}\, A_0$. Thus we see: $\boldsymbol{Re}\, A_0$ is not closed under uniform convergence on $|z| = 1$. In this paper, we shall show that this phenomenon is a special case of a very general property of algebras of functions.

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X. Let A be a complex linear subalgebra of $C(X)$ such that

(1) A is closed under uniform convergence;

(2) A contains the constant functions;

(3) A separates the points of X.

We write $\boldsymbol{Re}\, A$ for the set of functions $\boldsymbol{Re} f$ with f in A, that is, for the set of real parts of the functions in A. Clearly $\boldsymbol{Re}\, A$ is a (real) vector space of real-valued continuous functions on X. The purpose of this paper is to prove the following.

THEOREM. *If* $\boldsymbol{Re}\, A$ *is closed under uniform convergence, then* $A = C(X)$.

COROLLARY 1. *If* $\boldsymbol{Re}\, A$ *contains every real-valued continuous function on* X, *then* $A = C(X)$.

COROLLARY 2. (*Stone-Weierstrass*) *If* A *is closed under complex conjugation, then* $A = C(X)$.

Corollary 1 is an evident consequence of the theorem, and Corollary 2 follows upon observing that, if A is closed under complex conjuga-

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