ON ALMOST-COMMUTING PERMUTATIONS

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Suppose A and B are two permutations on a finite set X which commute on almost all of the points of X. Under what circumstances can we conclude that B is approximately equal to a permutation which actually commutes with A? The answer to this question depends strongly upon the order of the centralizer, $C(A)$, of A in the symmetric group on X; and this varies greatly according to the cycle structure of A, being comparatively small when A is either a product of few disjoint cycles or a product or a large number of disjoint cycles of different lengths and being comparatively large when A is a product of many disjoint cycles, all of the same length. We shall show by example that when the order of $C(A)$ is small there may exist a permutation B which commutes with A "almost everywhere" yet is not approximated by any element of $C(A)$. On the other hand, when A is a product of many disjoint cycles of the same length, we shall see that for any such permutation B, there must exist a permutation in $C(A)$ which agrees closely with B.

It is clear that if B is a permutation leaving fixed almost all points of X, then no matter what permutation A is given, B will commute with A on almost all points of X, and at the same time B can be closely approximated by an element of $C(A)$—namely, the identity. However, the examples we shall give will show that only when all (or nearly all) of the cycles of A are of the same length can we hope to approximate *every* B which nearly commutes with A by an element in $C(A)$. Accordingly, the bulk of this paper will be taken up with the study of the case in which A is a product of many disjoint cycles, all of the same length.

1. In order to get a satisfactory notation and a more compact way of discussing the problem, we begin by making the symmetric group $S_N(X)$ on the space X into a metric space. Here N denotes the cardinality of X, and it is to be understood that N is finite. Define, for any A in $S_N(x)$,

$$\| A \| = \frac{N - f_A}{N} \tag{1}$$

where f_A is the number of fixed points of A on X. Now define the distance $d(A, B)$ between two elements A and B of $S_N(X)$ to be

$$d(A, B) = \| AB^{-1} \| . \tag{2}$$

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