ON DIRECT SUMS AND PRODUCTS OF MODULES

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A well-known theorem of the theory of abelian groups states that the direct product of an infinite number of infinite cyclic groups is not free ([6], p. 48.) Two generalizations of this result to modules over various rings have been presented in earlier papers of the author ([3], [4].) In this note we exhibit a broader generalization which contains the preceding ones as special cases.

Moreover, it has other applications. For example, it yields an easy proof of a part of a result of Baumslag and Blackburn [2] which gives necessary conditions under which the direct sum of a sequence of abelian groups is a direct summand of their direct product. We also use it to prove the following variant of a result of Baer [1]: If a torsion group T is an epimorphic image of a direct product of a sequence of finitely generated abelian groups, then T is the direct sum of a divisible group and a group of bounded order. Finally, we derive a property of modules over a Dedekind ring which, for the ring Z of rational integers, reduces to the following recent theorem of Rotman [10] and Nunke [9]: If A is an abelian group such that $\mathrm{Ext}_Z(A, T) = 0$ for any torsion group T, then A is slender.

In this note all rings have identities and all modules are unitary.

1. The main theorem. Our discussion will be based on the following technical device.

DEFINITION 1.1. Let $\mathscr{F}$ be a collection of principal right ideals of a ring R. $\mathscr{F}$ will be called a *filter of principal right ideals* if, whenever aR and bR are in $\mathscr{F}$, there exists $c \in aR \cap bR$ such that cR is in $\mathscr{F}$.

We proceed immediately to the principal result of this note.

THEOREM 1.2. *Let $A^{(1)}, A^{(2)}, \cdots$ be a sequence of left modules over a ring R, and set $A = \prod_{i=1}^{\infty} A^{(i)}$, $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$. Let $C = \sum_\alpha \oplus C_\alpha$, where $\{C_\alpha\}$ is a family of left R-modules and α traces an index set I. Let $f\colon A \to C$ be an R-homomorphism, and denote by $f_\alpha\colon A \to C_\alpha$ the composition of f with the projection of C onto C_α. Finally, let $\mathscr{F}$ be a filter of principal right ideals of R. Then there exists aR in $\mathscr{F}$ and an integer $n > 0$ such that $f_\alpha(aA_n) \subseteqq \bigcap_{bR \in \mathscr{F}} bC_\alpha$ for all but a finite number of α in I.*

Proof. Assume that the statement is false. We shall first construct

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