FURTHER RESULTS ON p-AUTOMORPHIC p-GROUPS

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Graham Higman [3] has shown that a finite p-group, p an odd prime, with an automorphism permuting the subgroups of order p cyclically is abelian. In [1] a p-group was defined to be p-automorphic if its automorphism group is transitive on the elements of order p. It was conjectured that a p-automorphic p-group ($p \neq 2$) is abelian and proved that a counterexample must be generated by at least four elements. In this present paper we prove that a counterexample generated by n elements must be such that $n > 5$ and, if $n \neq 6$, then $p < n3^{n^2}$ (Theorem 3). We also show that the existence of a counterexample implies the existence of a certain algebraic configuration (Theorem 1). All groups considered are finite.

Notation. $\Phi(P)$ is the Frattini subgroup of the p-group P and P' is its commutator subgroup. $\Omega_i(P)$ is the subgroup generated by the elements of P whose orders do not exceed p^i. $Z(P)$ is the center of P. $F(m, n, p)$ denotes the set of p-automorphic p-groups P which enjoy the additional properties:

1. $P' = \Omega_1(P)$ is elementary abelian of order p^n.
2. $\Phi(P) = Z(P) = \Omega_m(P)$ is the direct product of n cyclic groups of order p^m.
3. $|P: \Phi(P)| = p^n$.

In [1] it was shown that a counterexample generated by n elements has a quotient group in $F(m, n, p)$. Hence, in arguing by contradiction, we may assume that a counterexample P is in $F(m, n, p)$.

Let $A = A(P) = \text{Aut } P$ and let $A_0 = \ker(\text{Aut } P \to \text{Aut } P/\Phi(P))$. Thus $A/A_0 = B$ is faithfully represented as linear transformations of $V = P/\Phi(P)$, considered as a vector space over $GF(p)$.

Since p is odd and $cl(P) = 2$, the mapping $\eta: x \to x^{p^m}$ is an endomorphism of P which commutes with each σ of Aut P. Since $\Omega_m(P) = \Phi(P)$, ker $\eta = \Phi(P)$, so η induces an isomorphism of V into $W = P'$. Since dim $V = \dim W$, η is onto.

The commutator function induces a skew-symmetric bilinear map of $V \times V$ onto W, (onto since P is p-automorphic) and since $\Phi(P) = Z(P)$, $(\ ,\ )$ is nondegenerate. Associated with $(\ ,\ )$ is a nonassociative product $\circ$, defined as follows: If $\alpha, \beta \in V$, say $\alpha = x\Phi(P)$, $\beta = y\Phi(P)$, then $[x, y]$ is an element of W which depends only on α, β, and so $[x, y] = z^{p^m}$ where the coset $\gamma = z\Phi(P)$ depends only on α, β. We write $\alpha \circ \beta = \gamma$. An immediate consequence of this condition is the statement that $\alpha \to \alpha \circ \beta$

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