ON UNIMODULAR MATRICES

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1. Introduction and summary. For the purpose of this note a matrix is called unimodular if every minor determinant equals 0, 1 or -1.

I. Heller and C. B. Tompkins [1] have considered a set

$$S = \{u_i, v_j, u_i + v_j, u_i - u_{i*}, v_j - v_{j*}\}$$

where the $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$ are linearly independent vectors in $m + n = k$-dimensional space E, and have shown that in the coordinate representation of S with respect to an arbitrary basis in E every nonvanishing determinant of k vectors of S has the same absolute value, and that, with respect to a basis in S, the vectors of S or of any subset of S are the columns of a unimodular matrix. For the purpose of this note the class of unimodular matrices obtained in this fashion shall be denoted as the class T.

A. J. Hoffman and J. B. Kruskal [4] have considered incidence matrices A of vertices versus directed paths of an oriented graph G, and proved that:

(i) if G is alternating, then A is unimodular;

(ii) if the matrix A of *all* directed paths of G is unimodular, then G is alternating. The terms are defined as follows. A graph G is oriented if it has no circular edges, at most one edge between any given two vertices, and each edge is oriented. A path is a sequence of distinct vertices $v_1, v_2, \cdots, v_k$ of G such that, for each i from 1 to $k-1$, G contains an edge connecting v_i with v_{i+1}; if the orientation of these edges is from v_i to v_{i+1}, the path is directed; if the orientation alternates throughout the sequence, the path is alternating. A loop is a sequence of vertices $v_1, v_2, \cdots, v_k$, which is a path except that $v_k = v_1$. A loop is alternating if successive edges are oppositely oriented and the first and last edges are oppositely oriented. The graph is alternating if every loop is alternating. The incidence matrix $A = (a_{ij})$ of the vertices v_i of G versus a set of directed paths $p_1, p_2, \cdots, p_k$ of G is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is in } p_j \\ 0 & \text{otherwise .} \end{cases}$$

The class of unimodular matrices thus associated with alternating graphs shall be denoted by K.

I. Heller [2] and [3] has considered unimodular matrices obtained

Received April 12, 1962. Research sponsored (for both authors) by the RAND Corporation, in conjunction with the Symposium in Combinatorial Analysis, and (for the first author) by the National Science Foundation.