BLOCK DIAGONALLY DOMINANT MATRICES AND GENERALIZATIONS OF THE GERSCHGORIN CIRCLE THEOREM

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1. Introduction. The main purpose of this paper is to give generalizations of the well known theorem of Gerschgorin on inclusion or exclusion regions for the eigenvalues of an arbitrary square matrix A. Basically, such exclusion regions arise naturally from results which establish the nonsingularity of A. For example, if $A = D + C$ where D is a nonsingular diagonal matrix, then Householder [7] shows that $\|D^{-1}C\| < 1$ in some matrix norm is sufficient to conclude that A is nonsingular. Hence, the set of all complex numbers z for which

$$\|(zI - D)^{-1}C\| < 1$$

evidently contains no eigenvalues of A. In a like manner, Fiedler [4] obtains exclusion regions for the eigenvalues of A by establishing the nonsingularity of A through comparisons with M-matrices.[1] Our approach, though not fundamentally different, establishes the nonsingularity of the matrix A by the generalization of the simple concept of a diagonally dominant matrix. But one of our major results (§ 3) is that these new exclusion regions can give significant improvements over the usual Gerschgorin circles in providing bounds for the eigenvalues of A.

2. Block diagonally dominant matrices. Let A be any $n \times n$ matrix with complex entries, which is partitioned in the following manner:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & & & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{bmatrix}, \tag{2.1}$$

where the diagonal submatrices $A_{i,i}$ are square of order n_i, $1 \leqq i \leqq N$. For reasons to appear in § 3, the particular choice $N = 1$ of

$$A = [A_{1,1}] \tag{2.1'}$$

will be useful. Viewing the square matrix $A_{i,i}$ as a linear transformation of the n_i-dimensional vector subspace Ω_i into itself, we associate with this subspace the vector norm $\|\boldsymbol{x}\|_{\Omega_i}$, i.e., if $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of

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[1] For the definition of an M-matrix, see § 4 or [8].