ANALYTIC METHODS IN THE STUDY OF ZEROS OF POLYNOMIALS

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Several analytic methods are used to obtain estimates for part or all zeros of a polynomial with complex coefficients and for linear combinations of polynomials. Some results of Biernacki, Montel and Specht are strengthened or generalized. Some results about the location of zeros of linear combinations of polynomials are also obtained.

1. Cauchy type estimates.

THEOREM 1. *Let* $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ *be a polynomial with complex coefficients. Let* $\beta'_2 \geqq \beta'_3 \geqq \cdots \geqq \beta'_n$ *be the ordered positive numbers* $|b_i| = |a_i \gamma^{-i}|, \gamma > 0, i = 2, \cdots, n$, *then all the zeros of the polynomial* $P(z)$ *are in the union of the two circles*:

$$|z| < \gamma(1 + \sigma_1) \text{ and } |z + a_1| \leqq \gamma$$

where

$$\sigma_1 = \beta'_2 - \frac{\delta'_2}{1 + \beta'_2} - \frac{\delta'_3}{(1 + \beta'_2)^2} - \cdots - \frac{\delta'_n}{(1 + \beta'_2)^{n-1}}$$

with

$$\delta'_i = \beta'_i - \beta'_{i+1}, \beta'_{n+1} = 0 .$$

Proof. It is well known (See e.g. [4]), that all zeros of the polynomial $P(z)$ are in the union of the two circles $|z + a_1| \leqq \gamma$ and $|z| \leqq \gamma(1 + \beta'_2)$. Let ζ be a zero of the polynomial $P(z)$. We may assume that $|\zeta| = \gamma r$, where $1 < r \leqq 1 + \beta'_2$. The inequality

$$|\zeta^n + a_1 \zeta^{n-1}| \leqq |a_2||\zeta|^{n-2} + \cdots + |a_n|$$

yields

$$(1) \quad r^{n-1}|\zeta + a_1| \leqq \gamma(|b_2|r^{n-2} + \cdots + |b_n|) \leqq \gamma(\beta'_2 r^{n-2} + \cdots + \beta'_n) ,$$

since β'_i, $i = 2, \cdots, n$, are decreasing and $r > 1$. Multiplying both sides of the inequality (1) by $(r-1)r^{-(n-1)}$ we get

$$(r-1)|\zeta + a_1| \leqq \gamma\left(\beta'_2 - \frac{\delta'_2}{r} - \cdots - \frac{\delta'_n}{r^{n-1}}\right) .$$

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