AN APPLICATION OF LINEAR PROGRAMMING TO PERMUTATION GROUPS

W. H. MILLS

Let S_N denote the symmetric group acting on a finite set X of N elements, $N \geqq 3$. Let σ and τ be elements of S_N. In a previous paper [1] the following question was raised: If σ and τ commute on most of the points of X, does it necessarily follow that τ can be approximated by an element in the centralizer $C(\sigma)$ of σ?

We define a distance $D(\sigma, \tau)$ between two elements σ and τ in S_N to be the number of points g in X such that $g\sigma \neq g\tau$. (This differs from the distance $d(\sigma, \tau)$ defined in [1] by a factor of N.) Then $D(\sigma\tau, \tau\sigma)$ is the number of points in X on which σ and τ do not commute. Let $D_\sigma(\tau)$ denote the distance from τ to the centralizer $C(\sigma)$ of σ in S_N. Thus

$$D_\sigma(\tau) = \min_{\lambda \in C(\sigma)} D(\tau, \lambda) .$$

It will be shown that the determination of $D_\sigma(\tau)$ is equivalent to the optimal assignment problem in linear programming.

The question raised in [1] can be phrased thus: If $D(\sigma\tau, \tau\sigma)$ is small, is $D_\sigma(\tau)$ necessarily small? If σ is not the identity we set

$$D_\sigma = \max_{\tau \notin C(\sigma)} D_\sigma(\tau)/D(\sigma\tau, \tau\sigma) .$$

Now D_σ is large unless σ is the product of many disjoint cycles, most of which have the same length. Some examples of this are worked out in detail in [1]. This leads us to study the case where σ is the product of m disjoint cycles of length n, where $N = nm$ and m is large. In [1] it was shown that if $m \geqq 2$, then

(a) if n is even, then $D_\sigma = n/4$, and

(b) if n is odd, $n \geqq 3$, then
$(n-1)/4 \leqq D_\sigma \leqq n/4$.

In the present paper it is shown that if n is odd, $n \geqq 3$, and $m \geqq n-2$, then

$$D_\sigma = (n-1)^2/(4n-6) .$$

1. Relation to linear programming. Let σ be an arbitrary element of the symmetric group S_N. We write σ as the product of disjoint cycles:

$$\sigma = C_1 C_2 \cdots C_m ,$$

Received October 22, 1962.