SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

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Let K be an arbitrary algebraic number field. We denote by n the degree of K, by $\mathfrak{f}$ an arbitrary ideal of K, by $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ prime ideals of K, by $\mu(\mathfrak{a})$ the Moebius function of the ideal $\mathfrak{a}$ of K, by $N\mathfrak{a}$ the norm of $\mathfrak{a}$, by $(\mathfrak{a}, \mathfrak{f})$ the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{f}$, and by $h(\mathfrak{f})$ the number of ideal classes H mod $\mathfrak{f}$. It is known that

$$(1)\qquad \begin{aligned} A(x, \mathfrak{f}) := \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} 1 &= \gamma(\mathfrak{f})x + R(x, \mathfrak{f}),\ R(x, \mathfrak{f}) = O(x^{1-1/n}) , \\ \gamma(\mathfrak{f}) &= \alpha \prod_{\mathfrak{p} \mid \mathfrak{f}} \left(1 - \frac{1}{N\mathfrak{p}}\right) \qquad (\alpha = \alpha(K) > 0) . \end{aligned}$$

According to [1], the proof of the generalized Selberg formula for ideal classes H mod $\mathfrak{f}$ in K:

$$(2)\qquad \sum_{\substack{N\mathfrak{p} \leqq x \\ \mathfrak{p} \in H \bmod \mathfrak{f}}} \log^2 N\mathfrak{p} + \sum_{\substack{N\mathfrak{p}\mathfrak{q} \leqq x \\ \mathfrak{p}\mathfrak{q} \in H \bmod \mathfrak{f}}} \log N\mathfrak{p} \log N\mathfrak{q} = \frac{2}{h(\mathfrak{f})} x \log x + O(x)$$

can be reduced to

$$(3)\qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^2 \frac{x}{N\mathfrak{a}} = \frac{2}{\gamma(\mathfrak{f})} \log x + O(1) ,$$

and (3) is established directly in [1]. First, we generalize (3):

THEOREM 1. *Let $r > 1$ be a rational integer; then*

$$\sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} = \frac{r}{\gamma(\mathfrak{f})} \log^{r-1} x + \sum_{t=1}^{r-2} c_t(r, \mathfrak{f}) \log^t x + O(1) ;$$

the constants $c_t(r, \mathfrak{f})$ resp. the constant in $O(1)$ depends on $K, r, t, \mathfrak{f}$ resp. $K, r, \mathfrak{f}$ only.

The formula

$$\sum_{\mathfrak{a} \mid \mathfrak{f}} \mu(\mathfrak{a}) = \begin{cases} 1 & \text{for } \mathfrak{f} = 1 , \\ 0 & \text{for } \mathfrak{f} \neq 1 \end{cases}$$

yields

LEMMA 1. *Let $f(x)$ be a complex valued function $(x \geqq 1)$; then*

Received July. 20, 1962. This work was supported by the National Science Foundation grant G-16305 to Purdue University.