SIMPLE PATHS ON POLYHEDRA

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In Euclidean d-space ($d \geqq 3$) consider a convex polytope whose $n\,(n \geqq d+1)$ vertices do not lie in a $(d-1)$-space. By the "path length" of such a polytope is meant the maximum number of its vertices which can be included in any single simple path, i.e., a path along its edges which does not pass through any given vertex more than once. Let $p(n, d)$ denote the minimum path length of all such polytopes of n vertices in d-space. Brown [1] has shown that $p(n, 3) \leqq (2n+13)/3$ and Grünbaum and Motzkin [3] have shown that $p(n, d) < 2(d-2)n^{\alpha}$ for some $\alpha < 1$, e.g., $\alpha = 1 - 2^{-19}$ and they have indicated how this last value may be improved to $\alpha = 1 - 2^{-16}$. The main object of this note is to derive the following result which, for sufficiently large values of n, represents an improvement upon the previously published bounds.

THEOREM.
$$p(n, d) < (2d+3)((1 - 2/(d+1))n - (d-2))^{\log 2/\log d} - 1 < 3d\, n^{\log 2/\log d}.$$

When $d = 3$ the example we construct to imply our bound is built upon a tetrahedron which we denote by G_0. Its 4 vertices, which will be called the 0th stage vertices, can all be included in a single simple path. Upon each of the 4 triangular faces of G_0 erect a pyramid in such a way that the resulting solid, G_1, is a convex polyhedron with 12 triangular faces. This introduces 4 more vertices, the 1st stage vertices, which can be included in a single simple path involving all 8 vertices of G_1. We may observe that it is impossible for a path to go from a 1st stage vertex to another 1st stage vertex without first passing through a 0th stage vertex.

The convex polyhedron G_2 is formed by erecting pyramids upon all the faces of G_1. Of the 12 2nd stage vertices thus introduced at most 9 can be included in any single simple path since, as before, no path can join two 2nd stage vertices without passing through an intermediate vertex of a lower stage and there are only 8 such vertices available.

The procedure continues as follows: the convex polyhedron G_k, $k \geqq 2$, is formed by erecting pyramids upon the 4.3^{k-1} triangular faces of G_{k-1}. Making repeated use of the fact that the method of construction makes it impossible for a path to join two vertices of the jth stage, $j \geqq 2$, without first passing through at least one vertex of a lower stage we find that at most 9.2^{j-2} of the 4.3^{j-1} vertices of the

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