POWERS OF A CONTRACTION IN HILBERT SPACE

SHAUL R. FOGUEL

Introduction. Let H be a Hilbert space and P an operator with $\|P\| = 1$. Our main problem is to find the weak limits of $P^n x$ as $n \to \infty$. This is applied to Markov Processes and to Measure Preserving Transformations.

Markov Processes. Let (Ω, Σ, μ) be a measure space. Let x_n be a sequence of real valued measurable functions on Ω and:

1. $\mu(x_{n+\alpha} \in A \cap x_{m+\alpha} \in B) = \mu(x_n \in A \cap x_m \in B)$.
2. *Conditional probability that $x_k \in A$ given x_i and x_j, $i < j < k$, is equal to conditional probability that $x_k \in A$ given x_j.*

Let $I(\sigma)$ denote the characteristic function of σ. Define $P(n)$ by linear extension of:

$P(n)\, I(x_0 \in A) =$ *Conditional probability that $x_n \in A$ given x_0.*

Then:

1′. $\|P(1)\| = 1$

2′. $P(n) = P(1)^n$.

For details see [1] and [2].

We will study limits of

$$(P(1)^n\, I(x_0 \in A),\, I(x_0 \in B)) = \mu(x_n \in A \cap x_0 \in B)\ .$$

Many of the results here appear in particular cases in [1,] [2] and [3].

1. Reduction to unitary operators. For every $x \in H$

a. $$\|P^{*k}P^kP^nx - P^nx\|^2 \leqq 2\|P^nx\|^2 - 2\,Re(P^{*k}P^kP^nxP^nx) = 2(\|P^nx\|^2 - \|P^{n+k}x\|^2) \underset{n\to\infty}{\longrightarrow} 0$$

b. $$\|P^kP^{*k}P^nx - P^nx\|^2 \leqq \|P^{*k}P^kP^{n-k}x - P^{n-k}x\|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore:

If weak lim $P^{n_i}x = y$ then $P^{*k}P^ky = P^kP^{*k}y = y$ (here and elsewhere n_i or m_i will denote a subsequence of the integers). This means $\|y\| = \|P^ky\| = \|P^{*k}y\|$. Notice that if $P^*Px = x$ then $\|Px\|^2 = (P^*Px, x) = \|x\|^2$. On the other hand

$$\|Px\|^2 = (P^*Px, x) \leqq \|P^*Px\|\,\|x\| \leqq \|x\|^2 \text{ since } \|P\| = 1.$$

Hence if $\|Px\| = \|x\|$ then $(P^*Px, x) = \|P^*Px\|\,\|x\|$ and thus $P^*Px = x$.

THEOREM 1.1. *Let* $K = \{x \mid \|P^kx\| = \|P^{*k}x\| = \|x\|\ k = 1, 2, \cdots\}$

Received November 26, 1962. The research reported in this document has been sponsored in part by Air Force Office of Scientific Research, O.A.R. through the European Office, Aerospace Research, United States Air Force.