ON COMPLEX APPROXIMATION

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1. Let C denote the set of complex numbers and G the set of Gaussian integers. In this note we prove the following theorem which is a two-dimensional analogue of Theorem 2 in [3].

THEOREM 1. *If* $\beta, \gamma \in C$, *then there exists* $u \in G$ *such that* $|\beta - u| < 2$ *and*

$$|\beta - u||\gamma - u| < \begin{cases} 27/32 & \text{if } |\beta - r| < \sqrt{11/8} \\ \sqrt{2}\,|\beta - \gamma|/2 & \text{if } |\beta - \gamma| \geq \sqrt{11/8}\,. \end{cases}$$

As an illustration of the application of Theorem 1 to complex approximation, we use it to prove the following result.

THEOREM 2. *If* $\theta \in C$ *is irrational and* $a \in C$, $a \neq m\theta + n$ *where* $m, n \in G$, *then there exist infinitely many pairs of relatively prime integers* $x, y \in G$ *such that*

$$|x(x\theta - y - a)| < 1/2\,.$$

The method of proof of Theorem 2 is due to Niven [6]. Also in [7], Niven uses Theorem 1 to obtain a more general result concerning complex approximation by nonhomogeneous linear forms.

Alternatively, Theorem 2 may be obtained as a consequence of a theorem of Hlawka [5]. This was done by Eggan [2] using Chalk's statement [1] of Hlawka's Theorem.

2. Theorem 1 may be restated in an equivalent form. For $u, b, c \in C$, define

$$g(u, b, c) = |u - (b + c)||u - (b - c)|\,.$$

Then Theorem 1 may be stated as follows.

THEOREM 1′. *If* $b, c \in C$, *then there exist* $u_1, u_2 \in G$ *such that*
(i) $|u_1 - (b + c)| < 2, |u_2 - (b - c)| < 2$
and for $i = 1,2$,

(ii) $$g(u_i, b, c) < \begin{cases} 27/32 & \text{if } |c| < \sqrt{11/32} \\ \sqrt{2}\,|c| & \text{if } |c| \geqq \sqrt{11/32}\,. \end{cases}$$

It is clear that Theorem 1′ implies Theorem 1 by taking

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