SOME DEGENERATE CAUCHY PROBLEMS WITH OPERATOR COEFFICIENTS

ROBERT CARROLL

1. Motivated in part by connections with problems in transonic gas dynamics there has been considerable interest in equations of the form

$$u_{tt} - K(t)u_{xx} + bu_x + eu_t + du - h = 0 \tag{1.1}$$

where d, b, e and h are functions of (x, t) (see here Bers [4] for a bibliography and discussion). In particular there arises the Cauchy problem for (1.1) in the hyperbolic region with data given on the parabolic line $t = 0$ (see in particular Protter [20], Conti [9], Bers [3], Berezin [2], Hellwig [12; 13], Frankl [10], Weinstein [25], Krasnov [15; 16], Carroll [8], Germain and Bader [11], and Barancev [1]). Protter assumes that $K(t)$ is a monotone increasing function of t, $K(0) = 0$, and shows that the Cauchy problem for (1.1) with initial data $u(x, 0)$ and $u_t(x, 0)$ prescribed on a finite x-interval, is correctly set (under suitable regularity assumptions) if $tb(x, t)/\sqrt{K(t)} \to 0$ as $t \to 0$. Thus in particular if $b \equiv 0$ the condition is automatically true. Krasnov considers generalized solutions and the equation

$$u_{tt} - \Sigma \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + \Sigma b_i \frac{\partial u}{\partial x_i} + e\frac{\partial u}{\partial t} + du = h\,. \tag{1.2}$$

Again the presence of first order terms b_i complicates the matter and (as with Protter for $K(t) \sim t^\alpha$) it is assumed that $b_i = O(t^{\alpha/2-1}\beta(t))$ where $\beta(t) \to 0$ (additional assumptions are also made). Krasnov supposes $\Sigma a_{ik}\xi_i\xi_k \geqq ct^\alpha \Sigma \xi_i^2$ with $h/t^{\frac{\alpha-1+\delta_0}{2}} \in L^2$ ($\delta_0 > 0$ is a number for which bounds are determined in the proof) and finds solutions u such that $u_t/t^{\frac{\alpha+1+\delta_0}{2}} \in L^2$ and $u_{x_i}/t^{\frac{1+\delta_0}{2}} \in L^2$. Thus the growth of h appears to play an important role in determining a solution in this more general equation (1.2). Slightly more general degeneracies for $\Sigma a_{ik}\xi_i\xi_k$ are mentioned by Krasnov but always in some comparison to a power of t.

It is one of the aims of the present paper to give a more precise estimate of the allowable degeneracy in relation to the growth of h and to give estimates for the solution. In particular we will not require that $K(t)$ be monotone. For simplicity we omit here first order terms in $\partial u/\partial x_i$; this will be dealt with, in an abstract framework, in a subsequent article. A summary of some of the present work was

Received March 26, 1962.