THE INVERSE OF THE ERROR FUNCTION

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1. Introduction. In a recent paper [3] J. R. Philip has discussed some properties of the function inverfc θ defined by means of

$$\theta = \operatorname{erfc}\ (\operatorname{inverfc}\ \theta)\ . \tag{1.1}$$

Since

$$\tfrac{1}{2}\pi^{1/2}(1 - \operatorname{erfc}\ x) = x - \frac{x^3}{3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \frac{x^9}{4!\,9} \cdots \tag{1.2}$$

it follows that

$$\operatorname{inverfc}\ \theta = u + \frac{1}{3}u^3 + \frac{7}{30}u^5 + \frac{127}{630}u^7 + \frac{4369}{22680}u^9 + \cdots\ , \tag{1.3}$$

where

$$u = \tfrac{1}{2}\pi^{1/2}(1 - \theta)\ .$$

The coefficients in (1.3) are rational numbers. It is therefore of some interest to look for arithmetic properties of these numbers.

It will be convenient to change the notation slightly. Put

$$f(x) = \int_0^{\infty} e^{-t^2/2}\, dt\ , \tag{1.4}$$

so that

$$f(x) = \left(\frac{\pi}{2}\right)^{1/2}(1 - \operatorname{erfc}\ 2^{1/2}x)$$

and let $g(x)$ denote the inverse function:

$$f(g(u)) = g(f(u)) = u\ , \tag{1.5}$$

where

$$g(u) = \sum_{n=0}^{\infty} A_{2n+1}\frac{u^{2n+1}}{(2n+1)!} \qquad (A_1 = 1)\ . \tag{1.6}$$

It follows from (1.4) and (1.5) that

$$g'(u) = \exp\left(\tfrac{1}{2}g^2(u)\right)\ . \tag{1.7}$$

Differentiating again, we get

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