A CHARACTERIZATION OF SCALAR TYPE OPERATORS ON REFLEXIVE BANACH SPACES

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Introduction. The main purpose of this paper is to characterize scalar operators on reflexive Banach spaces. This is accomplished in 4.2 and 4.4. However, most of the results are not limited to reflexive spaces.

We give a fundamental decomposition theorem for scalar operators in § 2, and show in § 3 that this decomposition is unique.

In what follows, all spaces are over the complex field, all Banach algebras have an identity of norm 1, and an operator will be a bounded linear transformation with range contained in its domain. This understanding will also cover material quoted from other sources.

1. Preliminaries. In this section we reproduce some machinery from [4] and [7] which will be needed in the sequel.

The definitions and results of this paragraph are taken from [4].

DEFINITION. Let X be a vector space. A semi-inner-product on X is a mapping $[\,,\,]$ of $X \times X$ into the field of complex numbers such that:

(i) $[x + y, z] = [x, z] + [y, z]$ for $x, y, z \in X$.

(ii) $[\lambda x, y] = \lambda[x, y]$ for $x, y \in X$, λ complex.

(iii) $[x, x] > 0$ for $x \neq 0$.

(iv) $|[x, y]|^2 \leqq [x, x][y, y]$.

We then call X a semi-inner-product space (abbreviated s.i.p.s.). If X is a s.i.p.s., then $[x, x]^{1/2}$ is a norm on X. On the other hand, every normed linear space can be made into a s.i.p.s. (in general, in infinitely many ways) so that the semi-innner-product is consistent with the norm—i.e., $[x,x]^{1/2} = \|x\|$, for each $x \in X$. By virtue of the Hahn-Banach theorem this can be accomplished by choosing for each $x \in X$ exactly one bounded linear functional f_x such that $\|f_x\| = \|x\|$ and $f_x(x) = \|x\|^2$, and then setting $[x, y] = f_y(x)$, for arbitrary $x, y \in X$.

DEFINITION. Given a linear transformation T on a s.i.p.s., we denote by $W(T)$ the set, $\{[Tx, x] \mid [x, x] = 1\}$, and call this set the numerical range of T.

An important fact concerning the notion of numerical range is the following:

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