# SOLUTION OF LOOP EQUATIONS BY ADJUNCTION

R. ARTZY

Solutions of integral equations over groups by means of adjunction of new elements have been studied by B. H. Neumann [3] and F. Levin [2]. Here an analogous question for loops will be dealt with, and the results will prove to be useful also for groups.

Let $(L, .)$ be a loop with neutral element $e$, $x$ an indeterminate. Let $w$ be a word whose letters are $x$ and elements of $L$. Let $n$ be the number of times that $x$ appears in $w$. Form $f(x)$ from $w$ by inserting parentheses between its letters so as to make it into a uniquely defined expression if juxtaposition means loop multiplication. The equation $f(x)=r$, $r$ in $L$, will be called *an integral loop equation in $x$ of degree $n$*.

An integral loop equation $f(x) = r$ is *monic* if $f(x)$ is a product of two factors both containing $x$. Every integral loop equation can be made monic by a finite number of left or right divisions by elements of $L$.

Not every integral loop equation has a solution as indicated by the monic example $x^2 = r$, $r \neq e$, $L$ the four-group. Our aim is finding a loop $E$ in which $L$ is embedded and in which $f(x) = r$ has a solution. The loop $E$ used here will be an extension loop [1] of $L$ by $(C_n, +)$, the cyclic group of order $n$. The construction follows the

*Extension Rule.* The elements of $E$ are ordered couples $(c, a)$ where $c \in C_n$, $a \in L$. Equality of couples is componentwise. The multiplication in $E$ is defined by $(c_1, a_1)(c_2, a_2) = (c_1 + c_2, a_1a_2 \cdot h(c_1, c_2))$, where $h(c_1, c_2)$ is an element of $L$ depending on $c_1$ and $c_2$, assuming the value $e$ except in the case when $c_1 + c_2 = 0$ and $c_1 \neq 0$.

THEOREM 1. *A monic integral loop equation $f(x) = r$ of degree $n$ over a loop $L$ has a solution in an extension loop $E = (C_n, L)$ constructed according to the Extension Rule, with $f(e)h(c, n - c) = r$ whenever $c \neq 0$.*

*Proof.* If the element $b$ of $L$ is represented in $E$ by $(0, b)$, $L$ is mapped isomorphically into $E$. Let $x$ be represented in $E$ by $(1, e)$, where 1 is a generator of $C_n$. All elements of $C_n$ will be written as integers. Then $f(x)$ can be constructed by stages. For every $x$ entering into the successive multiplication one summand 1 appears in the first component. In the second component only the loop elements of $f(x)$ will appear as factors because $x$ has the second component $e$. The $h$'s

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