THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA

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1. Introduction. Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X. We consider a closed subalgebra A of $C(X)$ which separates the points of X and contains the constants. We call A "a function algebra on X".

Let $Re\, A$ denote the class of functions u real and continuous on X such that for some f in A, $u = Re\, f$. Then $Re\, A$ is a real vector space of real continuous functions on X. What more can be said about $Re\, A$?

In [3] it was shown that $Re\, A$ cannot be closed under uniform convergence on X unless $A = C(X)$. Here we shall show that $Re\, A$ cannot be closed under multiplication unless $A = C(X)$. In other words:

Theorem 1: *If $Re\, A$ is a ring, then $A = C(X)$.*

This result was conjectured by K. Hoffman. As a corollary one gets the existence of a continuous function u on the unit circle having the following property: u has a continuous conjugate function (in the sense of Fourier theory) whereas u^2 does not. For we may take for A the algebra of continuous functions on the circle which extend analytically to the unit disk. Then $Re\, A$ is the class of all functions which are continuous and have continuous conjugates. But $A \neq C(X)$. Hence by Theorem 1, $Re\, A$ is not a ring, hence not closed under squaring, and so the desired u exists.

The existence of such a u had been shown in 1961 by J. P. Kahane (unpublished). It should be noted that if a function u is sufficiently smooth to have an absolutely convergent Fourier series, then u^2 does also, and hence u^2 does have a continuous conjugate.

2. The antisymmetric case. In this section we assume that A is anti-symmetric, i.e. contains no real functions except constants, and prove Theorem 1 under this hypothesis. This amounts to proving:

THEOREM 1′. *Let A be anti-symmetric and let $Re\, A$ form a ring. Then X consists of a single point.*

Assume X contains a point x_0 and another point x_1. We must deduce a contradiction. Fix u in $Re\, A$. Then (because of antisymmetry),

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