ON THE REPRESENTATION OF DOUBLY STOCHASTIC OPERATORS

JOHN V. RYFF

1. Introduction. A real n-square matrix is called *doubly stochastic* if it has nonnegative elements and each row and each column sums to 1. For any two n-vectors $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, let $x^* = (x_1^*, \cdots, x_n^*)$ and $y^* = (y_1^*, \cdots, y_n^*)$ be the vectors obtained from x and y by rearranging their respective components in nonincreasing order. Then for T doubly stochastic and $y = Tx$ it is not hard to show that

$$\sum_{i=1}^{k} y_i^* \leqq \sum_{i=1}^{k} x_i^* \qquad 1 \leqq k < n, \tag{1}$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i.$$

The converse is also true. If the inequalities (1) are valid for every pair of vectors related by the equation $y = Tx$, then T must be doubly stochastic. Employing a continuous version of (1), one can define on a space of integrable functions a class of linear operators which inherits many of the familiar properties of doubly stochastic matrices. A concrete representation of these operators will be obtained and a comparison made with the doubly stochastic operators defined by Rota [6].

In the following all functions and variables will be real. Measure will be understood to mean Lebesgue measure and will be denoted by μ. Sets will always be measurable and the term "almost everywhere" will usually be suppressed. $L^1 = L^1(0, 1)$ and $L^\infty = L^\infty(0, 1)$ are to have the conventional meaning as the spaces of integrable and essentially bounded "functions" on $[0, 1]$, with $[L^1]$ and $[L^\infty]$ to represent the operator spaces of L^1 and L^∞. Convergence is to be taken in the pointwise sense unless the contrary is indicated.

2. Rearrangements. If f is a measurable function on $[0, 1]$ consider

$$m(y) = \mu\{x : f(x) > y\}.$$

This function is nonincreasing, right-continuous and defined for all values of y. As such, the function m admits an inverse which will

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