CONJUGATE FUNCTIONS IN ORLICZ SPACES

ROBERT RYAN

1. The purpose of this paper is to prove the following results:

THEOREM 1. *Let*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan(1/2)t}\,dt = \lim_{\varepsilon\to+0}\left\{-\frac{1}{\pi}\int_\varepsilon^\pi\right\}.$$

The mapping $f \to \tilde{f}$ is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the L_p spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116–118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of $f \to \tilde{f}$. Our results are more general than Lozinski's results since we use the definition of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space L_1.

Section 2 contains preliminary material about Orlicz spaces. In § 3 we prove that boundedness implies reflexivity and in § 4 we prove the converse.

2. Let $v = \varphi(u)$ be a nondecreasing real valued function defined for $u \geqq 0$. Assume that $\varphi(0) = 0$, that φ is left continuous and that φ does not vanish identically. Let $u = \psi(v)$ be the left continuous inverse of φ. If $\lim_{u\to\infty}\varphi(u) = l$ is finite then $\psi(v) = \infty$ for $v > l$; otherwise $\psi(v)$ is finite for all $v \geqq 0$. The complementary Young's functions Φ and Ψ are defined by

$$\Phi(u) = \int_0^u \varphi(t)dt\,, \qquad \Psi(v) = \int_0^v \psi(s)ds\,.$$

Φ is an absolutely continuous convex function for $0 \leqq u < \infty$ and Ψ is absolutely continuous and convex in the internal where it is finite.

Received January 16, 1963. Sponsored by the Mathematics Research Center, U. S. Army Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.