SOME REMARKS ON FITTING'S INVARIANTS

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In the paper [2] Fitting introduced a sequence of ideals associated with a finitely generated module M over a commutative ring as follows: if $(E)\ 0 \to K \to F \to M \to 0$ is an exact sequence with F a free module on a basis $e(1), \cdots, e(n)$ and if $k(i) = \Sigma x(ij)e(j)$, i in some index set, generates K then the jth ideal $f(j; M)$ is generated by the $(n-j)x(n-j)$ determinants of the form $(x(uv))$. These ideals are independent of the sequence (E) and have the following properties:

(i) if h is a homomorphism from a ring R to a ring S and if M is a finitely generated R module then $S\cdot h(f(j\colon M))) = f(j; S \otimes_R M)$,

(ii) denoting by $ann(M)$ the annihilator of M we have $f(0\colon M) \leqq ann(M)$ and for sufficiently large m, $[ann(M)]^m \leqq f(0\colon M)$. Note also that $f(j; M) \leqq f(j+1; M)$ and that for j sufficiently large the ideals are all (1). In this paper we wish to make some remarks on the relation between these ideals and the concepts of flat and projective modules.

In the following we shall denote by $F(j; M)$ the R module $R/f(j; M)$ and by $F(M)$ the direct sum of the $F(j; M)$. We remark that the module $F(M)$ is finitely generated and it is free if and only if $F(j; M)$ is free (or zero) for each j. First note that for a free module N we have $F(s; N)$ is free for each s and that for any module (finitely generated) we may write $F(M) = R/f(0; M) \oplus \cdots \oplus R/f(s; M) \oplus \cdots$ where we suppose $f(r; M) \neq (1)$. If $F(j; M)$ is not free for some $j < r$ then $f(r; M) \neq (0)$ and hence $f(r-1\colon F(M)) = f(r; M)$ is neither (0) nor R.

THEOREM 1. *If M is a finitely generated module over a local ring R* (not necessarily noetherian) *then M is free if and only if $F(M)$ if free. If M is free and if I is the maximal ideal of R then*

$$\dim_{R/I}(R/I \otimes_R M) = \text{rank}\,(F(M)) = \text{rank}\,(M)\ .$$

Proof. If M is free then $F(M) = \Sigma_x F(x; M) = \Sigma_{x<n} R$ if M has rank n. Assume $F(M)$ is free and that $0 \to K \to F \to M \to 0$ is exact with F free over R. We may suppose that rank $(F) = \dim_{R/I}(R/I \otimes_R M)$ by the Nakayama lemma. Suppose, therefore, that $K \neq (0)$. Then $F(r-1; M)$, if the rank of F is r, has the form $\Lambda^r F/i(K) \wedge \Lambda^{r-1}F$ where i is the inclusion map of K into F and $\Lambda^r F$ denotes the homogeneous component of degree r in the Grassmann algebra of F.

Received December 6, 1961, and in revised form December 6, 1962. This work was supported in part by NSF Grant G-9508.