ON THE INTEGER SOLUTIONS OF $y(y+1)=x(x+1)(x+2)$

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This paper contains a solution of the following problem proposed to me by Professor Burton Jones to whom it was given by Mr. Edgar Emerson.

Problem. To show that the only integer solutions of

$$y(y+1) = x(x+1)(x+2) \tag{1}$$

are given by

$$x = 0, -1, -2, y = 0, -1; x = 1, y = 2, -3, x = 5, y = 14, -15 . \tag{2}$$

Put

$$2y + 1 = Y, \qquad 2x + 2 = X .$$

Then

$$2Y^2 = X^3 - 4X + 2 . \tag{3}$$

Obviously X in (3) cannot be odd so it must be shown that the only integer solutions of (3) are given by

$$X = 0, \pm 2, 4, 12 . \tag{4}$$

Diophantine equations of the form

$$Ey^2 = Ax^3 + Bx^2 + Cx + D \tag{5}$$

where A, B, C, D, E are integers are well known. I proved[1,2] in 1922 that the equation had only a finite number of integer solutions when the right hand side had no squared factor in x. In fact, this followed immediately from a result[3] I proved in 1913, by quoting Thue's result but which I did not know at that time. Finding these solutions may be a troublesome matter, involving many details, and usually rather difficult or even too difficult, to do.

One method requires a discussion of the field $R(\theta)$ defined by

$$A\theta^3 + B\theta^2 + C\theta + D \equiv 0 . \tag{6}$$

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[1] "Note on the integer solutions of the equation $Ey^2 = Ax^3 + Bx^2 + Cx + D$" Messenger of Math., **51** (1922), 169-171.

[2] "On the integer solutions of the equation $ey^2 = ax^3 + bx^2 + cx + d$" Proc. London Math. Soc., **21** (1923), 415-419.

[3] "Indeterminate Equations of the third and forth degrees" Quarterly Journal of Mathematics **45** (1914).