AN EXTENSION OF LANDAU'S THEOREM ON TOURNAMENTS

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In an ordinary (round-robin) *tournament* there are n people, $p_1, \cdots, p_n$, each of whom plays one game against each of the other $n - 1$ people. No game is permitted to end in a tie, and the *score* of p_i is the total number s_i of games won by p_i. By the *score sequence* of a given tournament is meant the set $S = (s_1, \cdots, s_n)$, where it may be assumed, with no loss of generality, that $s_1 \leqq \cdots \leqq s_n$. Landau [3] has given necessary and sufficient conditions for a set of integers to be the score sequence of some tournament. The object of this note is to show that these conditions are also necessary and sufficient for a set of real numbers to be the score sequence of a *generalized tournament*; a generalized tournament differs from an ordinary tournament in that as a result of the game between p_i and p_j, $i \neq j$, the amounts α_{ij} and $\alpha_{ji} = 1 - \alpha_{ij}$ are credited to p_i and p_j, respectively, subject only to the condition that $0 \leqq \alpha_{ij} \leqq 1$. The score of p_i is given by

$$s_i = \sum_{j=1}^{n}{}' \alpha_{ij} ,$$

where the prime indicates, for each admissible value of i, that the summation does not include $j = i$.

THEOREM. *A set of real numbers* $S = (s_1, \cdots, s_n)$, *where* $s_1 \leqq \cdots \leqq s_n$, *is the score sequence of some generalized tournament if and only if*

$$\sum_{i=1}^{k} s_i \geqq \binom{k}{2} , \tag{1}$$

for $k = 1, \cdots, n$ *with equality holding when* $k = n$.

Proof. The necessity of these conditions is obvious since (1) simply requires that the sum of the scores of any proper subset of the players be at least as large as the number of games played between members of this subset and that the sum of all the scores be equal to the total number of games played.

For terminology and results on flows in networks which will be used in the proof of the sufficiency of the above conditions see Gale [2]. A network N is constructed whose nodes are $x_1, \cdots, x_n$ and

Received January 23, 1963.