EXTREMAL ELEMENTS OF THE CONVEX CONE OF SEMI-NORMS

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1. Introduction. Let L be a real linear space and let p be a real function on L such that (1) $p(\lambda x) = |\lambda| p(x)$ for all x in L and all real λ, and $p(x_1 + x_2) \leqq p(x_1) + p(x_2)$ for all x_1 and x_2 in L, i.e. is a *semi-norm* on L. Since the sum of two semi-norms, $p_1 + p_2$ and the positive scalar multiplication of a semi-norm, λp, $\lambda > 0$ are semi-norms, the set of semi-norms on L, C form a convex cone. Those $p \in C$ such that if $p = p_1 + p_2$ where p_1 and $p_2 \in C$ we have p_1 and p_2 proportional to p are *extremal element* of C, [1]. In this paper it is shown that $p = |f|$, where f is a real linear functional of L is an extremal element of C. For L, the plane it is shown that these are the only extremal elements of C. Since norms are semi-norms, C includes this interesting class of functionals.

2. The main results. The convex cone C and the convex cone $-C$, the negatives of the elements of C have only the zero semi-norm in common since semi-norms are nonnegative. The zero semi-norm is an extremal element if one wishes to allow in the definition the vertex of a convex cone to be an extremal element. Below only the nonzero elements are considered.

The following lemma which characterizes the nature of certain semi-norms will be used in obtaining the two main theorems.

LEMMA 1. *If p is a semi-norm on L such that the co-dimension of $N(p) = 1$, then p is of the form $p = |f|$ where f is a linear functional on L.*

Proof. Let $b \in L \backslash N(p)$, where $N(p)$ is the *null space* of p. Then any element $x \in L$ can be written $x = z + \lambda b$ where $z \in N(p)$ and λ is real. Let $f(x) = \lambda p(b)$. Then clearly f is a linear functional on L. It shall now be shown that $|f(x)| = p(x)$ for all $x \in L$. Notice that

$$|f(x)| = |f(z + \lambda b)| = |\lambda p(b)| = |\lambda| p(b).$$

Thus

$$|f(x)| = p(\lambda b) = p(z) + p(\lambda b) \geqq p(z + \lambda b) = p(x).$$

The proof will be complete if it can be shown that the inequality

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