A NOTE ON UNCOUNTABLY MANY DISKS

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R. H. Bing has shown [2] that E^3 (Euclidean three dimensional space) does not contain uncountably many mutually disjoint wild 2-spheres. J. R. Stallings has given an example [6] to show that E^3 does contain uncountably many mutually disjoint wild disks. It is the goal of this note to show that E^3 does not contain uncountably many mutually disjoint disks each of which fails to lie on a 2-sphere in E^3. (A disk which fails to lie on a 2-sphere is necessarily wild.) For definitions the reader is referred to [1].

THEOREM 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists a disk D of the collection V such that D lies on a 2-sphere in E^3.*

The proof of Theorem 1 follows immediately from the following three lemmas.

LEMMA 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists an uncountable subcollection V^* of V such that if D belongs to V^*, x is an interior point of D, ax is an arc intersecting D only in the point x, and ε is a positive number then there exists an uncountable subcollection V_1 of V^* such that if D_1 is an element of V_1 then* (i) *$D_1 \cap ax \neq \phi$ and* (ii) *there is a homeomorphism of D_1 onto D which moves no point more than ε.*

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3. Let V' denote the subcollection of V defined as follows: D is an element of V' if and only if there exist a point x of Int D, an arc ax intersecting D only in x, and a positive number ε such that there is no uncountable subcollection V_1 of V such that if D_1 belongs to V_1 then (i) $D_1 \cap ax \neq \phi$ and (ii) there is a homeomorphism of D_1 onto D which moves no point more than ε.

It is clear that in order to establish Lemma 1 it is sufficient to show that the collection V' is countable. Suppose that V' is uncountable.

For each element D_α of V' let an arc a_α and a positive number ε_α be chosen such that (i) the common part of D_α and a_α is an endpoint of a_α which is on the interior of D_α, and (ii) a_α intersects only a countable number of elements D of V such that there is a homeomorphism of D onto D_α which moves no point by more than ε_α.

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