EQUALITY IN CERTAIN INEQUALITIES

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1. Introduction. Let $\sigma = (\sigma_1, \cdots, \sigma_n)$ be a point on the unit $(n-1)$-simplex S^{n-1}: $\sum_{i=1}^{n} \sigma_i = 1, \sigma_i \geqq 0$. Let $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ and $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n > 0$ be positive numbers and form the function on S^{n-1}

$$F(\sigma) = \sum_{i=1}^{n} \sigma_i \lambda_i \sum_{i=1}^{n} \sigma_i \mu_i \;. \tag{1.1}$$

The main purpose of this paper is to examine the structure of the set of points $\sigma \in S^{n-1}$ for which $F(\sigma)$ takes on its maximum value. In case a convex monotone decreasing function f is fitted to the points (λ_i, μ_i)(i.e. $f(\lambda_i) = \mu_i$), $i = 1, \cdots, n$, then it is not difficult to show that the maximum for $F(\sigma)$ on S^{n-1} is the upper bound given by M. Newman [4] in a recent interesting paper. In the case of the Kantorovich inequality [1] the function f is $f(t) = t^{-1}$, $\mu_i = \lambda_i^{-1}$, $i = 1, \cdots, n$. In this case a maximizing σ is $\sigma_1 = 1/2$, $\sigma_n = 1/2$, $\sigma_i = 0$, $i = 2, \cdots, n-1$, and if $\lambda_1 < \lambda_k < \lambda_n$, $k = 2, \cdots, n-1$, it is a corollary of our main result (Theorem 2) that this is the only choice possible for $\sigma \in S^{n-1}$ in order to achieve the maximum value.

We shall assume henceforth in this paper that $\mu_i = f(\lambda_i)$, $i = 1, \cdots, n$, where f is a monotone decreasing convex function defined on the closed interval $[\lambda_1, \lambda_n]$. In 2 we determine the structure of the set of $\sigma \in S^{n-1}$ for which $F(\sigma)$ is a maximum in the case in which f is assumed to be strictly convex. In 3 we investigate the structure of the set of unit vectors x for which the function

$$\varphi(x) = (Ax, x)(f(A)x, x) \tag{1.2}$$

assumes its maximum value on the unit sphere $\| x \| = 1$. Throughout, A is a positive definite hermitian transformation on an n-dimensional unitary space U with inner product (x, y). The eigenvalues of A are λ_i, $0 < \lambda_1 \leqq \cdots \leqq \lambda_n$, with corresponding orthonormal eigenvectors u_i, $Au_i = \lambda_i u_i$, $i = 1, \cdots, n$. Of particular interest in (1.2) is the choice $f(t) = t^{-p}$, $p > 0$.

Finally, in 4, we discuss the applications of the previous results to Grassmann compounds and induced power transformations associated with A. In two recent papers [2, 5] the Kantorovich inequality was applied to the compound to obtain inequalities involving principal subdeterminants of a positive definite hermitian matrix. We shall prove (Theorem 5) that these inequalities are in fact strict except in

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