FINITE-DIMENSIONAL PERTURBATION AND A REPRESENTATION OF SCATTERING OPERATOR

S. T. KURODA

1. Introduction. In what follows we shall be concerned with the problem of perturbation of continuous spectra by an operator of finite rank. Namely, we consider two self-adjoint operators H_0 and H_1 in a Hilbert space $\mathfrak{H}$ which are related to each other as follows:

$$(1.1)\qquad \begin{cases} H_1 = H_0 + V\,, \\ Vx = \sum_{k=1}^{r} c_k(x, \varphi_k)\varphi_k, \quad x \in \mathfrak{H}\,, \end{cases}$$

where $(\varphi_i, \varphi_j) = \delta_{ij}$ and c_k is a nonzero real number. For this problem the existence of the so-called wave operator $W_\pm$ and the scattering operator[1] S was proved by Kato [3] together with the unitary equivalence of the absolutely continuous (abbr. a.c.) parts of H_0 and H_1. As the first step of his proof, he considered the case of $r = 1$ in detail and proved a sort of explicit formulas for $W_\pm$ and S (or, in other words, representations of them in certain spectral representation spaces associated with H_0 and H_1[2]).

One of the main purposes of the present paper is to give a similar kind of formulas for $W_\pm$ and S in the case of an arbitrary finite value of r. For this purpose we use two kinds of spectral representation spaces. The first is the classical one due to Hellinger and Hahn (see [7, Chapt. VII]) and the second is one of its versions suitable for our problem.

For a self-adjoint operator H in $\mathfrak{H}$ and a finite subset $\{u_k; k=1, \cdots, n\}$ of $\mathfrak{H}$, we denote by $\mathfrak{L}(u_1, \cdots, u_n; H)$ the smallest closed subspace of $\mathfrak{H}$ containing $\{u_k\}$ and reducing H. Then, it is known [3, § 2] that $\mathfrak{H}_0 \equiv \mathfrak{L}(\varphi_1, \cdots, \varphi_r; H_0) = \mathfrak{L}(\varphi_1, \cdots, \varphi_r; H_1)$ and that $H_0 = H_1$ in the orthogonal complement $\mathfrak{H}_0'$ of $\mathfrak{H}_0$. As easy consequences, this yields that $W_\pm$ and S are reduced by $\mathfrak{H}_0'$ and that, on $\mathfrak{H}_0'$, they are simply the orthogonal projection on the a.c. subspaces of $\mathfrak{H}_0'$ with respect to H_0. Hence, in the study of a representation of $W_\pm$ and S, we can assume without essential loss of generality that

$$(1.2)\qquad \mathfrak{H} = \mathfrak{H}_0 = \mathfrak{L}(\varphi_1, \cdots, \varphi_r; H_p), \quad p = 0, 1\,.$$

Received November 14, 1962, and in revised form May 28, 1963. Research on this paper was partly supported by the National Science Foundation Grant GP-2.

[1] For the exact definition of $W_\pm$ and S, see (2.1) below.

[2] Certain similar representations were given in the scattering problems associated with partial differential operators of Schroedinger type. See, e.g. **[2]**.