GENERALIZED CONVEX INEQUALITIES

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This paper concerns certain cones of functions φ and their duals. The functions of these cones can be described globally by determinantal inequalities with the aid of certain auxiliary functions $\psi_1, \cdots, \psi_k$, or locally, if φ is sufficiently smooth, by differential inequalities. In the latter case smoothness hypotheses are imposed on $\psi_1, \cdots, \psi_k$ in order to define the relevant differential operator. The cones considered can be regarded as generalizations of the classical cone of convex functions. Various special cases occur in moment theory, problems of interpolation, differential equations, probability theory and elsewhere.

The elements of the corresponding dual cones are measures $d\mu$ (not necessarily nonnegative) which are orthogonal to $\psi_1, \cdots, \psi_k$, and in addition satisfy an inequality described with the aid of an integral operator (see (20) below). A sufficient condition for membership in the dual is that $d\mu$ have the minimal oscillation compatible with being orthogonal to $\psi_1, \cdots, \psi_k$, a condition depending on k but not otherwise on the choice of $\psi_1, \cdots, \psi_k$ (see Lemma B).

In the first section we consider the classical case of the cone of convex functions (and their discrete analogues, convex sequences). This case is presented in some detail, despite its familiarity and elementary character, in order to motivate the remainder. Much of this section was anticipated in [3] and elsewhere in the subsequent literature, but our viewpoint is somewhat different.

We introduce the general cones $C(\psi_1, \cdots, \psi_k)$ in § 3 where for ease of exposition we restrict ourselves to smooth $\psi_1, \cdots, \psi_k$. In this case the associated differential operator is easy to describe. We also impose strictness in certain constraining inequalities satisfied by $\psi_1, \cdots, \psi_k$. A characterization of the dual cone and the corresponding sufficient condition for membership in it are then given (Theorem 1 and Lemma B). As applications we list several known inequalities of previously unrelated character, and obtain some new ones. In § 5 we weaken some of the strictness conditions referred to above by introducing a useful procedure for approximating a cone with weakened hypotheses by one with strictness. In § 4 we exploit the translation invariant character of the cone whose associated differential operator is itself translation invariant (i.e. linear with constant coefficients).

Evidently we can weaken the hypothesis of smoothness concerning $\psi_1, \cdots, \psi_k$ and we can extend the considerations of § 4 to more gener-

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