QUANTIFIERS AND ORTHOMODULAR LATTICES

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1. Introduction. The "logic" of (non-relativistic) quantum mechanics is currently thought of as being the lattice of closed subspaces of a separable infinite dimensional Hilbert space [7, p. 49]. It has been speculated by P. Jordan [5] that this "logic" ought not to be a lattice at all, but rather what he calls a skew lattice. Given a lattice $L(\cap, \cup)$, Jordan observes that if one has functions $f, F: L \to L$ satisfying the conditions

$$(af \cup b)f = af \cup bf$$
$$af \leqq a$$
$$(a \cap bF)F = aF \cap bF$$
$$a \leqq aF ;$$

then $L(\wedge, \vee)$ is a skew lattice, where the operations $\wedge$, $\vee$ are defined by:

$$a \wedge b = a \cap bF$$
$$a \vee b = af \cup b .$$

Skew lattices themselves will not concern us here; rather we shall be interested in mappings on lattices having the above properties. Such mappings turn out to be generalizations of universal and existential quantifiers. With this thought in mind it seems of interest to begin an investigation of quantifiers on an orthomodular lattice, and in particular to consider the lattice $L(H)$ of closed subspaces of a Hilbert space H, determining all mappings f, F defined on $L(H)$ and satisfying Jordan's prescription.

The remaining part of this section will be devoted to a brief outline of known definitions and theorems that will prove useful in the sequel. These results can essentially be found in [1], [2], [9] and [10] but are included here for convenience. An *orthomodular lattice* is a lattice L with 0 and 1 equipped with an orthocomplementation $': L \to L$ and which satisfies the *orthomodular identity* $e \leqq f \Rightarrow f = e \vee (f \wedge e')$. *Henceforth L will always represent an orthomodular lattice.* If $e, f \in L$ with $e \leqq f$ it is easily shown that the interval $L(e, f) = \{g \in L: e \leqq g \leqq f\}$ is itself an orthomodular lattice with orthocomplementation

$$g \to g^{\#} = e \vee (f \wedge g') = (e \vee g') \wedge f .$$

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