THE METHOD OF INTERIOR PARALLELS APPLIED TO POLYGONAL OR MULTIPLY CONNECTED MEMBRANES

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1. Introduction.

1.1. The scope of this paper is (a) to discuss the possibilities of the method of interior parallels (*Makai, Pólya, Payne-Weinberger*) by considering the case of polygonal membranes (§ 2); (b) to extend it to multiply connected domains in a more satisfactory manner than has hitherto been proposed (§ 3); to this end we use a result of H. F. Weinberger [7] on the existence of an "effectless cut", published immediately after the present paper.

1.2. We consider the problem of a vibrating membrane covering a plane domain G and fixed along the boundary Γ. We are interested in the first eigenvalue λ_1 of the problem $\Delta u + \lambda u = 0$ in G, $u = 0$ along Γ; by Rayleigh's principle,

$$\lambda_1 \leqq R[v] \equiv \frac{D(v)}{\iint_G v^2 dA} \qquad \text{if } v = 0 \text{ along } \Gamma .$$

$dA = dxdy$ is the element of area; $D(v) = \iint_G grad^2 v dA$, Dirichlet's integral; $R[v]$, Rayleigh's quotient.

The method of interior parallels consists in using trial functions v whose level lines are parallel to Γ. It was first introduced by E. Makai [2, 3]: using the trial function $v(Q) = \delta_{Q\Gamma}$ ($Q \in G$, δ = Euclidean distance), he obtained, for every simply or doubly connected membrane G of area A, fixed along its boundary Γ of total length L_Γ, the bound

$$\lambda_1 \leqq 3\frac{L_\Gamma^2}{A^2} . \tag{1}$$

His proof makes use of B. Sz.-Nagy's [6] inequality

$$q(\delta) \leqq L_\Gamma \tag{2}$$

bounding the total length $q(\delta)$ of the "interior parallel at distance δ" in a simply or doubly connected domain; as Sz.-Nagy proved, this length exists for almost all values of δ.

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