ON A CLASS OF EQUIVALENT SYSTEMS OF LINEAR INEQUALITIES

I. HELLER

0. Summary. This note is concerned with the question of when a matrix A of mn columns and rank $m+n-1$ is transformable into a matrix of a special class known as the constraint matrices of an m by n transportation program. The question is of practical significance in the solution of linear programs. The main result, a set of necessary and sufficient conditions on the matrix A, is formulated in § 3 (Theorem 3.3), and proved in §§ 4–5. As application, § 6 outlines a method of testing for the conditions of the theorem and of effectuating the transformation when the conditions are satisfied.

1. Introduction. A finite system of linear inequalities can, in general, be reexpressed in the form

$$\mathrm{A}x = b,\ x \geqq 0. \tag{1.6}$$

The objective is to characterize among the systems (1.6) those that are equivalent (in a sense to be defined in § 2) to the systems occurring as constraints of a special type of linear optimization programs, known heuristically as "transportation" programs, which admit relatively simple and efficient algorithms of solution (see for instance Dantzig [1] and [2] and Ford and Fulkerson [3]).

We shall refer to a system of the form

$$\sum_{i=1}^{m} \varepsilon_{ij}x_{ij} = c_j, \quad \sum_{j=1}^{n} \varepsilon_{ij}x_{ij} = r_i, \qquad x_{ij} \geqq 0 \tag{1.7}$$

where $\varepsilon_{ij} = \pm 1$ $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$, as "the constraints of a transportation program," the special case $\varepsilon_{ij} = 1$ representing the constraints of a "standard" transportation program.

Interpreting $x = (x_{ij})$ as vector in R^{mn}, (1. 7) can also be written in the form

$$Dx = c,\ x \geqq 0 \tag{1.8}$$

where $c' = (r_1, r_2, \cdots, r_m,\ c_1, c_2, \cdots, c_n)$, hence, $c \in R^{m+n}$ and D is a matrix of $m+n$ rows and mn columns d_ν $(\nu = 1, 2, \cdots, mn)$, also denotable as d_{ij} $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$ in lexicographic order, with

Received August 13, 1962. This work was supported by National Science Foundation Grant 20035 at Stanford University