ON THE ASYMPTOTIC INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Various methods have been employed for the asymptotic integration of ordinary differential equations, e.g., successive approximations (cf. [2]), topological arguments involving Waiewski's or similar principles (cf. [4], [5], [8]), and fixed points theorems (cf. [3]). The object of this note is illustrate the application for this purpose of a simple and general theorem which is based, on the one hand, on Massera and Schäffer's [7] use of the open mapping theorem and, on the other hand, on Tychonoff's fixed point theorem. This general theorem is essentially a corrected version of a theorem of Corduneanu [1].

Below $x, y, \cdots$ are elements of a finite dimensional Banach space X of norms $\|x\|, \|y\|, \cdots$. L denotes the space of real-valued functions $\varphi(t)$ on J: $0 \leqq t < \infty$ with the topology of convergence in the mean L^1 on bounded intervals. B denotes a Banach space of real-valued functions $\varphi(t)$ on $0 \leqq t < \infty$, norm $|\varphi|_B$, which is stronger than L (in the sense that B is contained in L algebraically and convergence in B implies convergence in L; [7]). Examples of such spaces are $L^p = L^p(0, \infty)$, $1 \leqq p \leqq \infty$, with norm $|\varphi|_p$ or the subspace L_0^∞ of L^∞ of functions $\varphi(t)$ satisfying $\varphi(t) \to 0$ as $t \to \infty$.

$L(X), L^p(X), B(X), \cdots$ will represent the space of measurable functions $x(t)$ from J to X such that $\varphi(t) = \|x(t)\|$ is in $L, L^p, B, \cdots$. In the case L^p or B, the norm $|\varphi|_p$ or $|\varphi|_B$ will be abbreviated to $|x|_p$ or $|x|_B$. $C(X)$ is the space of continuous functions from J to X with the topology of uniform convergence on bounded intervals.

Consider a homogeneous and an inhomogeneous system of linear differential equations

$$x' = A(t)x\,, \tag{1.1}$$

$$x' = A(t)x + g(t)\,, \tag{1.2}$$

in which $g(t) \in L(X)$, $A(t)$ is an endomorphism of X for fixed t and is locally integrable on J. If $\mathscr{D}$ is a Banach space stronger than $L(X)$, a $\mathscr{D}$-solution $x(t)$ of (1.1) or (1.2) is a solution $x(t) \in \mathscr{D}$. Let

Received October 31, 1962, and in revised form December 27, 1962. The research of the first author was supported by the Air Force Office of Scientific Research. It was completed while the second author was a Fellow of the John Simon Guggenheim Memorial Foundation.

The authors wish to thank J. J. Schäffer for calling their attention to the paper of Corduneanu [1].