CONVERGENCE OF EXTENDED BERNSTEIN POLYNOMIALS IN THE COMPLEX PLANE

J. J. GERGEN, F. G. DRESSEL, AND W. H. PURCELL, JR.

1. Introduction. Let $f(x)$ be defined on $[0, 1]$. The following two theorems on the Bernstein polynomials corresponding to f,

$$B_n(x; f) = \sum_{\lambda=0}^{n} f\left(\frac{\lambda}{n}\right)\binom{n}{\lambda} x^{\lambda}(1-x)^{n-\lambda}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

are well known.

THEOREM I. *If $f(x)$ is continuous on $[0, 1]$, then $B_n(x; f) \to f(x)$ as $n \to \infty$ uniformly on $[0, 1]$.*

THEOREM II. *If $f(z)$, $z = x + iy$, is analytic in the interior E of the ellipse with foci at $z = 0$ and $z = 1$, then $B_n(z; f) \to f(z)$ as $n \to \infty$ on E, this convergence being uniform on each closed subset of E.*

The first of these results is due to S. Bernstein [1], the second to L. V. Kantorovitch [6] (See also [4], [7]).

For $f(x)$ defined on $[0, \infty)$ the functions

$$P_k(x; f) = e^{-kx} \sum_{\lambda=0}^{\infty} \frac{(kx)^{\lambda}}{\lambda!} f\left(\frac{\lambda}{k}\right), \qquad 0 < k, \tag{1.2}$$

form a natural extension of the Bernstein polynomials, the terms of (1.2) corresponding to a Poisson distribution in much the same manner as the terms of (1.1) correspond to a binomial distribution. The functions (1.2) have been considered by Favard [5], Szász [9], and Butzer [3] for the real case. The results of Favard and Szász include the following analogue of Theorem I.

THEOREM III. *If $f(x)$ is continuous on $[0, \infty)$, and if $f(x) = O(x^A)$* [Szász], *or more generally, if $f(x) = O(e^{Ax})$* [Favard] *as $x \to \infty$, where A is a positive, real constant, then $P_k(x; f) \to f(x)$ as $k \to \infty$ for x on $[0, \infty)$, this convergence being uniform on each finite subinterval of $[0, \infty)$.*

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