ENTROPY AND SINGULARITY OF INFINITE CONVOLUTIONS

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Introduction. Let $\Phi(\omega)$ be a random variable which takes only a finite number of values

$$x_1, x_2 \cdots, x_p ,$$

with probabilities

$$\pi_1, \pi_2, \cdots, \pi_p .$$

Let $A(x)$ be the distribution function of $\Phi(\omega)$.

We shall be concerned here with infinite convolutions of the type

$$F(x, r) = A\left(\frac{x}{r_1}\right) * A\left(\frac{x}{r_2}\right) * \cdots * A\left(\frac{x}{r_n}\right) * \cdots \tag{I.1}$$

where $r = (r_1, r_2, \cdots, r_n, \cdots)$ is a given sequence of non-vanishing real numbers. From standard theorems of Probability theory it follows that the convolution product in (I.1) converges (if we exclude the trivial case $p = 1$, $x_1 = 0$) if and only if $\sum r_n^2 < \infty$ and either

$$E(\Phi) = 0 \tag{I.2}$$

or

$$E(\Phi) \neq 0 \text{ but } \sum r_n \text{ is convergent.} \tag{I.3}$$

In either case, the limit distribution $F(x, r)$ is continuous and pure.[1] A proof of this result in the case that $\Phi(\omega)$ takes only the two values ± 1 with equal probabilities can be found in [4].

We can and shall restrict our study to the case $E(\Phi) = 0$. Our main result here concerns the distributions $F(x, r)$ generated by sequence $\{r_n\}$ such that, for some $0<\beta<1$,

$$r_n = 0[\beta^n] . \tag{I.4}$$

Under this hypothesis it is easy to see that for a given $A(x)$, when β is sufficiently small, $F(x, r)$ is necessarily singular. This result follows from the simple fact that the set of points of increase of $F(x, r)$, for all sufficiently small β, has zero measure. On the other hand, as β increases towards one $F(x, r)$ in general will become

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[1] That is either absolutely continuous or purely singular.