INTEGRAL EQUATIONS IN NORMED ABELIAN GROUPS

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1. Introduction. Suppose Z is an additive abelian group with additive identity element N and a "norm" $\|\cdot\|$ such that $\|N\| = 0$, and if $z, w \in Z$, then $\|z + w\| \leqq \|z\| + \|w\|$, $\|-z\| = \|z\|$, and $\|z\| > 0$ unless $z = N$. Suppose furthermore that Z is complete with respect to the metric induced by this norm. Let B denote the set of all transformations from Z into Z. Suppose $[a, b]$ is a closed number interval, $A \in Z$, and each of F and G is a function from $[a, b]$ into B.

Under suitable restrictions on F and G, we wish to find a function Y from $[a, b]$ into Z satisfying the integral equation

$$Y(x) = A + \int_a^x dG \cdot FY\,, \tag{1.1}$$

where FY denotes the function from $[a, b]$ into Z defined by $[FY](x) = F(x)Y(x)$. Notice that parentheses are used in denoting the image of a number, but not in denoting the image of an element of B. We wish to express a solution of (1.1) as a product integral

$$Y(x) = \pi_a^x(1 + dG \cdot F)A\,. \tag{1.2}$$

The terms "integral" and "product integral" will be defined in the next section, but the notation is quite suggestive, taking $1z = z$ for $z \in Z$.

A related problem has been treated by J. W. Neuberger [1]. Let us perform a "change of variable." That is, let R denote the function from $[a, b]$ into B defined by $R(x)z = \int_a^x dG \cdot Fz$, where Fz denotes the function from $[a, b]$ into Z defined by $[Fz](x) = F(x)z$. Then (1.1) becomes, at least formally

$$Y(x) = A + \int_a^x dR \cdot Y\,. \tag{1.3}$$

Under suitable restrictions, Neuberger expresses solutions of (1.3) as the product integral

$$Y(x) = \pi_a^x(1 + dR)A\,, \tag{1.4}$$

or, in Neuberger's notation

$$Y(x) = {}_a\pi^x(T, A)\,, \qquad T(p, q) = 1 + R(p) - R(q)\,.$$

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