CONVERGENT SOLUTIONS OF ORDINARY LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS

W. J. A. CULMER AND W. A. HARRIS, JR.

1. Introduction. This paper is concerned with the difference equation

$$x(s+1) = A(s)x(s) \tag{1.1}$$

where the 2 by 2 matrix $A(s)$ has a convergent series representation

$$A(s) = s^h \sum_{k=0}^{\infty} A_k s^{-k}\ , \qquad |s| > s_0 \tag{1.2}$$

and $A_0 \neq 0$. The A_k are constant matrices, the independent variable s is complex, h is a constant, and $x(s)$ is a column vector. We seek two independent vector solutions or a fundamental solution to the corresponding matrix equation

$$X(s+1) = A(s)X(s)\ .$$

The number of linearly independent solutions is not apparent since for both $A(s) \equiv 0$ and $A(s) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (1.2) has only $x(s) \equiv 0$ as a solution. We will show that there will be two linearly independent solutions unless the determinant of $A(s)$ vanishes identically, in which case there will either be one or none.

We begin, in § 2, by reducing the matrix $A(s)$ to one of eight canonical forms which (after factoring out $s^{h'}$) have convergent expansions in s^{-1} or $s^{-1/2}$.

In § 3 we construct formal solutions for these difference equations by substitution and direct comparison. The formal solutions will contain in general divergent power series, but it is expected that these formal solutions are asymptotic representations of true solutions in appropriate regions of the s-plane. Section 4 is devoted to estimates on the growth of the coefficients in these formal series.

In § 5 we consider integral equations for vector functions $w(t)$ whose Laplace transforms are simply related to certain formal series occurring in our formal solutions. Using the estimates of § 4 (in all but one case) we obtain true solutions of these integral equations.

These vector functions $w(t)$ are used in §§ 6 and 7 to construct true solutions of the original difference equation. Theorems of Doetsch and Nörlund are utilized to prove that the formal solutions obtained

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