HOMOMORPHISMS OF NON-COMMUTATIVE $*$-ALGEBRAS

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1. Introduction. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and ν a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. This paper is a study of the continuity properties of ν which depend only on the structure of $\mathfrak{A}$; $\mathfrak{B}$ is completely arbitrary. The algebras considered are non-commutative.

If ν is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then the function $|x| = \|\nu(x)\|$, $x \in \mathfrak{A}$, is a multiplicative semi-norm on $\mathfrak{A}$. Conversely, every multiplicative semi-norm on $\mathfrak{A}$ arises from a homomorphism in this way. Thus all results on continuity of homomorphisms can be stated in terms of multiplicative semi-norms.

Section 2 contains material concerning units in $\mathfrak{A}$ and $\mathfrak{B}$ and the relation between homomorphisms and multiplicative semi-norms.

Section 3 is devoted to the proof of the main technical device of the paper: If $\{g_n\}$ and $\{f_n\}$ are sequences in $\mathfrak{A}$ with $g_n g_m = 0$, $n \neq m$, and $f_n g_m = 0$, $n \neq m$, then, under any homomorphism ν of $\mathfrak{A}$ into a Banach algebra $\mathfrak{B}$, the sequence $\{\|\nu(f_n g_n)\|/\|f_n\|\|g_n\|\}$ is bounded.

In § 4 the separating ideals for ν in $\mathfrak{A}$ and $\mathfrak{B}$ are defined and several of their properties are exhibited. The separating ideal $\mathscr{S}$ for ν in $\mathfrak{A}$ is the set of x in $\mathfrak{A}$ for which there is a sequence $\{x_n\}$ in $\mathfrak{A}$ with $x_n \to 0$ and $\nu(x_n) \to \nu(x)$. An application of the main boundedness theorem (Theorem 3.1) shows that if $\{x_n\}$ is a sequence in $\mathscr{S}$ with $x_n x_m = 0$, $n \neq m$, then $\nu(x_n)^3 = 0$ for all but a finite number of n.

In § 5 we restrict attention to the case in which ν is an isomorphism and $\mathfrak{A}$ is a B^* algebra. In this case $\mathscr{S}$ is the zero ideal. This fact enables us to show that there is a constant M such that $\|x\| \leqq M\|\nu(x)\|$, $x \in \mathfrak{A}$. This result is analogous to an important theorem of Kaplansky [4]: any multiplicative norm on the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space majorizes the supremum norm. A theorem due to Bonsall [2] implies the following similar result: if $|\cdot|$ is a multiplicative norm on the algebra $\mathfrak{A}$ of bounded operators on a Banach space, there is a constant β such that for $T \in \mathfrak{A}$, $\|T\| \leqq \beta|T|$, where $\|\cdot\|$ is the usual operator norm. Although our result is similar, our approach is quite different. Kaplansky's proof depends heavily on commutativity; Bonsall's on the existence of nonzero finite dimensional operators which, of course, are not necessarily present in an arbitrary B^* algebra. Notice that if $\mathfrak{A}$ is a Banach algebra with the property that for every

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