ENTROPIES OF SEVERAL SETS OF REAL VALUED FUNCTIONS

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Introduction. In this paper the entropies of several sets of real valued functions are calculated. The entropy of a metric set, a notion introduced by Kolmogorov [2], is a measure of its size in terms of the minimal number of sets of diameter not exceeding 2ε necessary to cover it. The most striking use of this notion to date has been given by Kolmogorov [4] and Vituškin [7] who have shown that not all functions of n variables can be represented by functions of fewer variables if only functions satisfying certain smoothness conditions are allowed. For an exposition of this and other topics related to entropy see [5]. For other entropy calculations by the present author see [1]. The Kolmogorov-Vituškin result makes use of the following entropy calculation:

Let $F^n_{q=p+\alpha}(C, K) = F^n_q$ denote the class of real valued functions $f(x) = f(x_1, \cdots, x_n)$ defined on the unit cube S_n in the Euclidean n space which satisfy $|f(x)| \leqq C$ and have all partial derivatives of the order $k \leqq p$, with the pth order derivatives satisfying a Lipschitz condition of order α, $0 < \alpha \leqq 1$, with Lipschitz constant K:

$$|f^{(p)}(x) - f^{(p)}(x')| \leqq K|x - x'|^\alpha , \quad x, x' \in S_n .$$

Under the uniform metric ρ,

$$\rho(f, g) = \max_{x \in S_n} |f(x) - g(x)| ,$$

Kolmogorov [4, Th. XIV, p. 308] obtains

$$(1) \qquad H_\varepsilon(F^n_q) \asymp (1/\varepsilon)^{n/q} .$$

(The various symbols are defined below). In particular, with $p = 0$ and $n = 1$, this reads

$$(2) \qquad H_\varepsilon(\mathrm{Lip}_K \alpha) \asymp (1/\varepsilon)^{1/\alpha} ,$$

where we have written $\mathrm{Lip}_K \alpha$ in place of F^1_α.

The object of this paper is first to generalize (2) to sets of functions which satisfy a smoothness condition (§ 1), and second to show that

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