A TOPOLOGICAL MEASURE CONSTRUCTION

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1. Introduction. The purpose of this paper is to give two results in topological measure theory that generalize two well known results for metric spaces.

The principal one of these, which is given in § 3, concerns the construction of a measure from a nonnegative set function. Carathéodory [1] has done this in a natural way in defining Carathéodory linear measure in a finite dimensional Euclidean space. It is well known that this Carathéodory construction can be applied in metric spaces to produce measures for which the open sets are measurable.[1] Our treatment produces a measure for which the open F_σ sets, in a regular topological space, are measurable, and is identical with the Carathéodory measure in case the topology is metrizable. Since each open set in a metric space is an F_σ this provides a generalization of the metric result.

Our other result, which is given in § 2, concerns a necessary and sufficient condition for the measurability of open sets. A well known condition for this in metric spaces is that the measure be additive on any two sets which are a positive distance apart. When this condition is changed to require the additivity on two sets whose closures do not intersect, it becomes a necessary and sufficient condition for the measurability of the open F_σ sets, for a normal topological space. We show that the condition of normality can be weakened to one of "ϕ Normal" (see definition 2.4.5 below). Since a metric space is normal and therefore ϕ Normal, and since each open set of a metric space is an F_σ, this provides a clear generalization of the metric result.

At first glance the weakened normality condition of 2.4.5 appears to add little to topological measure theory. However, this is just the condition that results from our construction in § 3 (even though the topology is not necessarily normal) and hence the results of § 2 help us to obtain the results of § 3.

Nowhere in this paper is an assumption of local compactness made.

2. Conditions for measurability.

2.1. DEFINITIONS.

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[1] See, for example, Method II, page 105, of [**3**].