BOUNDS FOR EIGENVALUES AND GENERALIZED CONVEXITY

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1. Introduction. The eigenvalue problem associated with a vibrating string which is under unit tension has a nonnegative integrable density function ρ defined for $x \in [a, b]$ and which is held elastically at its ends is

$$u'' + \lambda\rho(x)u = 0 \,, \qquad u'(a) - h_a u(a) = u'(b) + h_b u(b) = 0 \tag{1.1}$$

where $0 \leqq h_a, h_b \leqq \infty$. The limiting case $h_a \to \infty$, $h_b \to \infty$ corresponds to the fixed end point problem. In general, the eigenvalues of the system (1.1) are nonnegative, simple and depend on the function ρ. We denote them accordingly by

$$0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots < \lambda_n[\rho] < \cdots \,.$$

We consider the problem of finding uniform upper and lower bounds for $\lambda_n[\rho]$ $(n = 1, 2, \cdots)$ when ρ is restricted to belong to a specific set of functions. In particular, we consider sets of functions ρ which are either convex or concave in the following generalized sense.

Let

$$L(y) \equiv (r(x)y')' - p(x)y \,, \qquad x \in [a, b] \tag{1.2}$$

where r and p are real-valued continuous functions on $[a, b]$ with $r(x) > 0$ and $r \in C'$ on $[a, b]$. Furthermore, we consider only those equations of the form (1.2) whose solutions satisfy the

EXISTENCE PROPERTY. *There exists a unique solution y of* (1.2) *through the points (x_1, y_1), (x_2, y_2) where $a \leqq x_1 < x_2 \leqq b$ and y_1, y_2 are arbitary real numbers. We denote the values of y by $y(x) = y(x;\ x_1, y_1;\ x_2, y_2)$.*

DEFINITION. A real function ρ is sub-(L) on $[a, b]$ if for arbitrary x_1, x_2 such that $a \leqq x_1 < x_2 \leqq b$, we have

$$\rho(x) \leqq y(x;\ x_1, \rho(x_1);\ x_2, \rho(x_2)) \,, \quad x \in [x_1, x_2] \,.$$

ρ is super-(L) if

$$\rho(x) \geqq y(x;\ x_1, \rho(x_1);\ x_2, \rho(x_2)) \,, \quad x \in [x_1, x_2] \,.$$

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