ON THE SPECTRUM OF A TOEPLITZ OPERATOR

HAROLD WIDOM

Given a function $\phi \in L_\infty(-\pi, \pi)$, the Toeplitz operator T_ϕ is the operator on H_2 (the set of $f \in L_2$ with Fourier series of the form $\sum_0^\infty c_n e^{in\theta}$) which consists of multiplication by ϕ followed by P, the natural projection of L_2 onto H_2: if $f \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}$ then $Pf \sim \sum_0^\infty c_n e^{in\theta}$. Succinctly,

$$T_\phi f = P(\phi f) \qquad f \in H_2.$$

In [5] a necessary and sufficient condition on ϕ was given for the invertibility of T_ϕ. This will be stated below. (The paper [5] is needlessly complicated. In a recent paper of Devinatz [1], however, all results of [5] and more are proved without undue complication in a general Dirichlet algebra setting.) Halmos [2] has posed the following as a test question for any theory of invertibility of Toeplitz operators: *Is the spectrum of a Toeplitz operator necessarily connected?* We shall shown here that the answer is *Yes.*

The proof consists mainly of applications of Theorem I of [5], which says the following.

A necessary and sufficient condition for the invertibility of T_ϕ is the existence of function ϕ_+ and ϕ_- belonging respectively to H_2 and $\bar{H}_2$ (the set of complex conjugates of H_2 functions) such that

(a) $\phi = \phi_+\phi_-$,

(b) $\phi_+^{-1} \in H_2$ *and* $\phi_-^{-1} \in \bar{H}_2$,

(c) *for* $f \in L_\infty$, $Sf = \phi_+^{-1} P \phi_-^{-1} f \in L_2$, *and* $f \to Sf$ *extends to a bounded operator on* L_2.

We don't want to prove the theorem here but we do have to say where the functions $\phi_\pm$ come from under the assumption that T_ϕ is ivertible. If we set

$$\psi_+ = T_\phi^{-1} 1, \ \bar{\psi}_- = T_\phi^{*-1} 1$$

then it can be shown that $\phi\psi_+\psi_- = c$, a constant. We must have $c \neq 0$ since $\psi_\pm$ can vanish only on sets of measure zero and ϕ is not identically zero. One then defines

$$\phi_+ = 1/\psi_+, \quad \phi_- = c/\psi_-$$

and (a) and (b) hold.

As for the relevance of condition (c), it turns out that the ex-

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