ON A CLASS OF SINGULAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH A NON LINEAR PARAMETER

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Introduction. Singular second order differential equations with a non-linear parameter play an important role in mathematical physics. For instance, the radial wave equation for a relativistic particle with zero spin in a centrally symmetric potential field, $V(r)$, the Klein-Gordon equation:

$$(1) \qquad u'' + \frac{2}{r}u' + \left\{\left[\frac{E + eV(r)}{\hbar c}\right]^2 - \frac{l(l+1)}{r^2} - \left(\frac{Mc}{\hbar}\right)^2\right\}u = 0$$

$$[' \equiv d/dr]$$

contains the energy parameter E in a nonlinear fashion. If $V(r) = aZ/r$ is a Coulomb field the transformation $x = 2(1 - E^2)^{1/2}r$ will transform equation (1) into a Whittaker equation

$$(2) \qquad \frac{d^2u}{dx^2} + \left[-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - m^2}{x^2}\right]u = 0$$

where

$$k = E\alpha Z(1 - E^2)^{-1/2}$$

and

$$m = [(l + 1/2)^2 - \alpha^2 Z^2]^{1/2} .$$

The eigenvalues are the roots of a transcendental equation [1]

$$\frac{B - n\pi}{\lambda} - \frac{1}{\lambda}\arg\left[\Gamma\left(\frac{1}{2} - \frac{E\alpha Z}{(1 - E^2)^{1/2}}\right) + i(\alpha^2 Z^2 - (l + 1/2)^2)^{1/2}\right] = \log(1 - E^2)^{1/2} .$$

For more general potentials than the Coulomb potential the spectral properties of a second order differential equation with a non-linear parameter are more difficult to obtain. For second order equations with a linear parameter the analytical methods of Weyl [7] and Titchmarsh [5] for the Schrödinger equation

$$(3) \qquad x''(r) + [\lambda - U(r)]x(r) = 0$$

may be used. These methods have been extended by Titchmarsh [6],

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