ON THE SOLVABILITY OF NONLINEAR FUNCTIONAL EQUATIONS OF 'MONOTONIC' TYPE

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1. Introduction. Let X and Y be a conjugate pair of reflexive Banach spaces (with real or complex scalars) such that X has a smooth unit ball. For $x \in X$, $y \in Y$, we denote the natural pseudo-inner-product by $\langle x, y\rangle$. Let $f\colon X \rightarrow Y$ be a continuous monotonic function—i.e., one satisfying $Re\langle x_1 - x_2, f(x_1) - f(x_2)\rangle \geqq 0$ for all $x_1, x_2 \in X$. The main object of this paper is to present a theorem on the solvability of the equation $f(x) = u$, for given $u \in Y$, analogous to the ordinary "intermediate-value theorem" for a continuous (monotonic!) real-valued function of a real variable. In finite-dimensions, the known result [1] is that the range R of f is an almost-convex set (contains the interior of its convex hull, where "interior" may be taken relative to the smallest real flat containing R—see below for the definitions).

In order to preserve, so far as possible, duality between the domain and the range of f, theorems will be proved first on "monotonic" subsets of the product-space $X \times Y$, and then afterwards applied to the graph of f.

The theorems of this paper result from an attempt to obtain the same general kind of theorems as one gets by the "variational method", as developed especially by E. H. Rothe, without assuming that f is the Fréchet differential of a real scalar function. In the variational theory, the assumption of monotonicity of f turns up in the form of convexity of the associated scalar, which in turn guarantees weak lower-semicontinuity. (In order to see the connection, compare Theorem 6 of [3] and Theorem 4.2 of [6]).

2. Preliminaries. Let X be a Banach-space and Y its conjugate-space, or vice versa. In $X \times Y$, we define the M-relation (as in [1], [3]) by: $(x_1, y_1)M(x_2, y_2)$ provided $Re\langle x_1 - x_2, y_1 - y_2\rangle \geqq 0$. A set $E \subset (X \times Y)$ is called *monotonic* provided each pair of points of E is M-related, and is called *maximal* if it cannot be embedded in a properly larger monotonic subset of $X \times Y$. If, for any (x_1, y_1) and $(x_2, y_2) \in E$ we have $Re\langle x_1 - x_2, y_1 - y_2\rangle = 0$ implies $x_1 = x_2$ and $y_1 = y_2$, then E will be called *strictly* monotonic.

Note that $\langle x, y\rangle$ is a bilinear form rather than a sesquilinear form, that is, for α complex, $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$. Nevertheless, in the theorems of this paper, it may be thought of as the usual inner product when $X = Y = H$, where H is a Hilbert space, because the

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