ON THE MONOTONICITY OF THE GRADIENT OF A CONVEX FUNCTION

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The object of this note is to present some elementary theorems concerning convex functions in n-dimensions and, more generally, topological vector spaces. These theorems are all essentially generalizations of the theorem "the derivative of a convex function of one real variable is monotonic non-decreasing", and appear to have been overlooked in the literature.

Let X be a topological vector space with real scalars, and Y the conjugate-space (space of continuous linear functionals) of X. We shall write $y(x)$, for $x \in X$, $y \in Y$, as $\langle x, y \rangle$ to facilitate applications to Hilbert space. The convex (real-valued) function ϕ will always be presumed to have convex domain $D \subset X$, and satisfies the inequality

$$\phi(sx_1 + tx_2) \leqq s\phi(x_1) + t\phi(x_2)$$

for all x_1, x_2 in D, all $s \geqq 0, t \geqq 0, s + t = 1$. The *graph* G of ϕ is a subset of the topological vector space $X \dotplus R$, and it is obvious that the "set of points lying above the graph of ϕ": $A = \{(x, r): x \in D, r \geqq \phi(x)\}$ is a convex set. (This condition is also sufficient for the convexity of ϕ.)

DEFINITION 1. A set $E \subset X \times Y$ is called a monotonic set provided that, for all (x_1, y_1) and (x_2, y_2) in E, $\langle x_1 - x_2, y_1 - y_2 \rangle \geqq 0$.

DEFINITION 2. ([6]) For $D \subset X$, a function $F: D \rightarrow Y$ is called monotonic provided the graph of F is a monotonic set. Now, it is well known that the conjugate space of $X \dotplus R$ is $Y \dotplus R$, and that a closed hyperplane in $X \dotplus R$ is of the form $\{(x, r): \langle x, y_0 \rangle + rr_0 = \alpha\}$ for some $y_0 \in Y, r_0 \in R, \alpha \in R$. (See [2], p. 26, Théorème 1.) This representation is non-unique, but if $r_0 \neq 0$, the equation $\langle x, y_0 \rangle + rr_0 = \alpha$ can be solved for r, and the resulting equation is, in an obvious sense, unique. These facts motivate the following definition:

DEFINITION 3. A gradient hyperplane H of ϕ is a closed hyperplane of support to A, the set of points lying above the graph of ϕ in $X \dotplus R$, such that H can be written in the form $\{(x, r): r = \phi(x_0) + \langle x - x_0, y_0 \rangle\}$. (Note the analogy with the first two terms of a Taylor-series for ϕ.)

REMARK 1. This definition might be considered inappropriate if ϕ is not everywhere-defined over X; this problem will not concern us here.

Received February 11, 1963.