ON FINITE SUMS OF RECIPROCALS OF DISTINCT nTH POWERS

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Introduction. It has long been known that every positive rational number can be represented as a finite sum of reciprocals of distinct positive integers (the first proof having been given by Leonardo Pisano [6] in 1202). It is the purpose of this paper to characterize (cf. Theorem 4) those rational numbers which can be written as finite sums of reciprocals of distinct nth *powers* of integers, where n is an arbitrary (fixed) positive integer and "finite sum" denotes a sum with a finite number of summands. It will follow, for example, that p/q is the finite sum of reciprocals of distinct squares[1] if and only if

$$\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

Our starting point will be the following result:

THEOREM A. *Let n be a positive integer and let H^n denote the sequence $(1^{-n}, 2^{-n}, 3^{-n}, \cdots)$. Then the rational number p/q is the finite sum of distinct terms taken from H^n if and only if for all $\varepsilon > 0$, there is a finite sum s of distinct terms taken from H^n such that $0 \leqq s - p/q < \varepsilon$.*

Theorem A is an immediate consequence of a result of the author [2, Theorem 4] together with the fact that every sufficiently large integer is the sum of distinct nth powers of positive integers (cf., [8], [7] or [3]).

The main results. We begin with several definitions. Let $S = (s_1, s_2, \cdots)$ denote a (possibly finite) sequence of real numbers.

DEFINITION 1. $P(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \varepsilon_k s_k$ where $\varepsilon_k = 0$ or 1 and all but a finite number of the ε_k are 0.

DEFINITION 2. $Ac(S)$ is defined to be the set of all real numbers x such that for all $\varepsilon > 0$, there is an $s \in P(S)$ such that $0 \leqq s - x < \varepsilon$. Note that in this terminology Theorem A becomes:

$$P(H^n) = Ac(H^n) \cap Q \tag{1}$$

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[1] This result has also been obtained by P. Erdös (not published).