SOME APPLICATIONS OF MEANS OF CONVEX BODIES

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Let A be a real, positive definite, $n \times n$ matrix; with A we associate, in the Euclidean n-space R_n, the ellipsoid $E(A)$ of points x for which

$$(x, Ax) \leqq 1$$

where (x, y) denotes the usual inner product. In references [5], [6], [7] certain means of convex bodies were studied. It will be shown here that two particular means of ellipsoids of the type $E(A)$ correspond to two simple combinations of the corresponding matrices A. The applications mentioned in the title rest upon this correspondence. The first two give results about positive definite matrices, including a refinement of a determinant inequality of Minkowski; the third application shows the existence of a set of unique ellipsoids related to a convex body by a set of similar extremal problems, the classical Loewner ellipsoid being a particular instance.

Throughout this paper the letters A and B, sometimes with distinguishing marks, denote real, positive definite, $n \times n$ matrices. The distance from x to the origin is written $\|x\|$.

1. The distance and support functions of $E(A)$ are:

$$F(x) = \sqrt{(x, Ax)}\ , \qquad H(x) = \sqrt{(x, A^{-1}x)}\ .$$

In the first case, if $x \neq 0$, we have $F(x) = \|x\|/\|z\|$ where $x/\|x\| = z/\|z\|$ and $(z, Az) = 1$, and so

$$\begin{aligned} \|x\|/\|z\| &= \|x\| \sqrt{(z/\|z\|, Az/\|z\|)} \\ &= \|x\| \sqrt{(x/\|x\|, Ax/\|x\|)} = \sqrt{(x, Ax)}\ . \end{aligned}$$

In the second case

$$H(x) = \max_y\, (x, y) \qquad \text{where} \qquad (y, Ay) = 1\ .$$

We represent y in the form $\lambda A^{-1}x + v$ where $(x, v) = 0$. Then

$$(y, Ay) = \lambda^2 (x, A^{-1}x) + (v, Av)\ ,$$

whence

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