INFINITE PRODUCTS OF ISOLS

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Introduction. In [1] the notion of the sum of an infinite number of isols is introduced. In this paper we shall similarly attack the problem of the product of an infinite number of isols. Before proceeding to this it is necessary to review the concept of exponentiation. Let $\varepsilon = \{0, 1, \cdots\}$ be the set of nonnegative integers. f *is a finite function* if $\delta f = \varepsilon$ (δf, ρf are domain, range of f respectively) and $\{x : f(x) \neq 0\}$ is finite. The set $\{x : f(x) \neq 0\}$ is called the *essential domain* of f (denoted $\delta_e f$) and the set $\{f(x) : f(x) \neq 0\}$ the *essential range* of f (denoted $\rho_e f$). f is a *finite function from the set* β *into the set* α if $\delta_e f \subseteqq \beta$ and $\rho_e f \subseteqq \alpha$. It can be shown (cf. [3], 181) that there exists a recursive function $r_n(x)$ to two variables such that

(1) All finite functions are generated without repetitions in the sequence $\{r_n(x)\}$.

(2) From n, one can effectively find $r_n(x)$.

(3) From $r_n(x)$, one can effectively find n. Then for any subsets α and β of ε we define

$$\alpha^\beta = \{n : r_n(x) \text{ is a finite function from } \beta \text{ into } \alpha\} .$$

In case α and β are finite it is necessary that $0 \in \alpha$ in order to make α^β have m^n elements where α has m elements and β has n elements. If $A \neq 0$ we let $A^B = \mathrm{Req}(\alpha^\beta)$ where $0 \in \alpha \in A, \beta \in B$. Otherwise $0^B = 1$ if $B = 0, 0^B = 0$ if $B > 0$.

Let $R = \mathrm{Req}(\varepsilon)$. It is known (cf. [3], 189) that $2^R = R$. Since we would like an infinite product of identical factors to reduce to an exponentiation, we see that an infinite product of isols may not be an isol. On the other hand, if X is an infinite isol, then so is 2^X (cf. [3], 182). Thus depending on which exponent we use to formalize the concept of an infinite product of repeated factors we may or may not obtain an isol.

A one-to-one function t_n from ε into ε is *regressive* (cf. [1]) if there is a partial recursive function $p(x)$ such that $\rho t \subseteqq \delta p$ and $p(t_0) = t_0, (\forall n)\, (p(t_{n+1}) = t_n)$. A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. There is no loss of generality by also supposing that p has the following additional properties: $\rho p \subseteqq \delta p$ and $(\forall x)\, (x \in \delta p \rightarrow (\exists n)\, (p^{n+1}(x) = p^n(x)))$ (superscript denotes iterate). Define p^* by $\delta p^* = \delta p$ and $p^*(x) = (\mu n)\, (p^{n+1}(x) = p^n(x))$. Define $\bar{p}$ by $\delta \bar{p} = \delta p$ and $\rho_{\bar{p}(x)} = \{p(x), \cdots, p^n(x)\}$ where $n = p^*(x)$. Two one-to-one

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