ENDOMORPHISMS OF FUNCTION-SPACES WHICH LEAVE STABLE ALL TRANSLATION-INVARIANT MANIFOLDS

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0. The problem. Preliminary definitions

0.1. Throughout §§ 0–5 of this paper X denotes a separated locally compact group with left Haar measure dx. Most of the time we shall consider a topological vector space F of complex-valued functions, measures or distributions on X which is invariant under left [right] translations $\tau_a\,[\rho_a]\,(a \in X)$. By a left [right] invariant manifold in F we shall mean a closed, left [right] translation invariant vector subspace of F. Our theorems result from an attempt to derive a general representation formula for those endomorphisms T of F which have the property of leaving stable left [right] invariant manifolds in F, i.e. which are such that $T(M) \subset M$ for each invariant manifold M in F. This demand is tantamount to requiring that $Tf \in M_f$ for each $f \in F$, where M_f is the left [right] invariant manifold in F generated by F. M_f is the intersection of all left [right] invariant manifolds in F which contain f; and, if each translation operator $\tau_a[\rho_a]$ is a continuous endomorphism of F, M_f is simply the closure in F of the set of all finite linear combinations of translates $\tau_a f[\rho_a f]$ of f.

In most cases one may broaden the problem by demanding merely that Tf be the limit in some specified sense of finite linear combinations of $\tau_a f[\rho_a f]$, this sense being not necessarily derived from the initial topology of F. In fact, no topology may be involved which is related to F in particular.

The type of representation theorem which appears in those cases which have been handled successfully asserts that Tf must take the form of a convolution $\mu * f\,[f * \mu]$, where μ is some measure or distribution on X depending only upon T. Hidden within this conclusion are the facts that T commutes with right [left] translations; and, in some cases at least, that T is automatically continuous. On the other hand, there are cases where we have found it necessary to assume continuity of T at the outset, so that the role of this hypothesis is not too clear.

I am indebted to a referee for pointing out to me a result due to P. Eymard [2], which asserts that, if X is any Lie group, and if T belongs to the von Neumann algebra of operators on $L^2(X)$ generated by left translations, then there exists a Schwartz distribution μ on X such that $Tf = \mu * f$ for each $f \in C_c^\infty(X)$. This result is closely related to Theorem 4.1 *infra*.

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