ON THE SPECTRUM OF A TOEPLITZ OPERATOR

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1. Introduction. A Toeplitz operator T is one which transforms a sequence $x = (x_0, x_1, x_2 \cdots)$ into a sequence y according to the formal law

$$(1) \qquad \sum_{k=0}^{\infty} c_{n-k} x_k = y_n , \qquad n = 0, 1, 2, \cdots .$$

If the complex coefficients c_n satisfy the condition

$$(2) \qquad \sum_{n=-\infty}^{\infty} |c_n| < \infty ,$$

then T carries each l_p space $(1 \leqq p \leqq \infty)$ into itself. Here l_p is the Banach space of all complex sequences $x = (x_0, x_1, \cdots)$ for which the norm

$$\| x \|_p = \left\{ \sum_{n=0}^{\infty} |x_n|^p \right\}^{1/p}$$

is finite. As usual, $\| x \|_\infty = \sup |x_n|$.

Under the assumption (2), M. G. Krein [7] has described the spectrum of T as an operator in l_p. His method uses some rather deep theorems on the factorization of absolutely convergent Fourier series. Actually, Krein's emphasis is on the Wiener-Hopf integral operator, which is the continuous analogue of T. Without knowledge of Krein's work, Calderón, Spitzer, and Widom [4] used similar methods to obtain most of the same results on the spectrum of T.

The key to the spectrum of T, in any l_p space, is the continuous closed curve Γ defined by

$$(3) \qquad \lambda = F(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} , \qquad 0 \leqq \theta < 2\pi .$$

For any point $\lambda \notin \Gamma$, it is the winding number of Γ about λ which alone determines the exact spectral character of λ. The precise results, which are due to Krein, will be stated below. Since the spectrum is always a closed set, it follows from these results that the entire curve Γ belongs to the spectrum of T. There remains, however, the finer question: for what reason is a point $\lambda \in \Gamma$ in the spectrum? That is, to which part of the spectrum does λ belong? For operators T satisfying

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