NORMAL FORM FOR A PFAFFIAN

RICHARD ARENS

1. Introduction. It is well known that "generally" (which is to is the say, usually) a Pfaffian, or 1-form,

$$\alpha = a_1(x)dx^1 + \cdots + a_n(x)dx^n$$

in $\boldsymbol{R}^n$ has one or the other of the two representations

1.1 $$\alpha = u^1du^2 + \cdots u^{2p-1}du^{2p} + \begin{cases} 0 \\ du^{2p+1} \end{cases}$$

in an appropriate coordinate system $(u^1, u^2, \cdots, u^n)$. Moreover, the last index ($2p$ or $2p+1$) appearing in 1.1 is the rank r of the $n \times (n+1)$ matrix

1.2 $$\begin{bmatrix} a_1 & a_2 & \cdots a_n \\ a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots x_{2n} \\ \cdot & & \\ \cdot & & \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{bmatrix},$$

in which a_{ij} is an abbreviation for $\partial a^i/\partial x^i - \partial a_j/\partial x^i$.

It goes without saying that this is regarded as a *local* proposition, indicating that if the rank of 1.2 were constant in some neighborhood of a point $\boldsymbol{P}_0$, then a smaller neighborhood of $\boldsymbol{P}_0$ and a curvilinear coordinate system valid on that neighborhood, could be found yielding the representation 1.1.

It is very probable that a satisfactory proof concerning the possibility of reducing a Pfaffian in this way exists in the literature[1]. Nevertheless, it should be pointed out that the accepted version is not exactly true (and this is part of our object in writing this paper.)

Consider the Pfaffian $ydx + 2xdy$ in ordinary $\boldsymbol{R}^2$. The Pfaffian matrix is

$$\begin{bmatrix} y & 2x \\ 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

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[1] For references to the older literature, see pp. 324-6 of E. Weber's article in the Encyklop. d. Math. Wiss. Band II, Erster Teil, Erste Hälfte (1.I) Teubner (1916). This article attributes to Frobenius a proof of the sort of proposition stated above, which we will therefore call the *accepted* version.