ON THE RING-LOGIC CHARACTER OF CERTAIN RINGS

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Introduction. Boolean rings $(B, \times, +)$ and Boolean logics (= Boolean algebras) $(B, \cap, ^*)$ though historically and conceptionally different, are equationally interdefinable in a familiar way [6]. With this equational interdefinability as motivation, Foster introduced and studied the theory of ring-logics. In this theory, a ring (or an algebra) R is studied modulo K, where K is an arbitrary transformation group in R. The Boolean theory results from the special choice, for K, of the "Boolean group," generated by $x^* = 1 - x$ (order 2, $x^{**} = x$). More generally, let $(R, \times, +)$ be a commutative ring with identity 1, and let $K = \{\rho_1, \rho_2, \cdots\}$ be a transformation group in R. The *K-logic* (*or K-logical algebra*) of the ring $(R, \times, +)$ is the (operationally closed) system $(R, \times, \rho_1, \rho_2, \cdots)$ whose class R is identical with the class of ring elements, and whose operations are the ring product "$\times$" of the ring together with the unary operations $\rho_1, \rho_2, \cdots$ of K. The ring $(R, \times, +)$ is called a *ring-logic*, mod K if (1) the "+" of the ring is *equationally* definable in terms of its K-logic $(R, \times; \rho_1, \rho_2, \cdots)$, and (2) the "+" of the ring is *fixed* by its K-logic. Of particular interest in the theory of ring-logics is the *normal group* D which was shown in [1] to be particularly adaptable to p^k-rings. Our present object is to extend further the class of ring-logics, modulo the normal group D itself. A by-product of this extension is the following result, namely, any finite commutative ring with zero radical is a ring-logic, mod D (see Corollary 8). Furthermore, in Corollary 10, we prove that, more generally, any (not necessarily finite) ring with unit which satisfies $x^n = x (n$ fixed, $\geqq 2)$ is a ring-logic (mod D). Finally, we compare the normal group with the so-called *natural* group in regard to the ring-logic character of a certain important class of rings (see section 3).

1. The finite field case. Let $(F_{p^k}, \times, +)$ be a Galois (finite) field with exactly p^k elements (p prime). Then, as is well known, F_{p^k} contains a multiplicative generator, ξ;

$$F_{p^k} = \{0, \xi, \xi^2, \cdots, \xi^{p^k-1} (=1)\} .$$

We now have the following (compare with [1]).

THEOREM 1. *Let F_{p^k} be a Galois field, and let ξ be a generator of F_{p^k}. Then the mapping $x \to x^\frown$ defined by*

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